Some Remarks on the Conditional Independence and the Markov Property

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1. Introduction

Recently, there has been some work on stochastic differential equations with boundary conditions (cf., for instance, [1,2,7,8,9]). This has been possible thanks to the development of the extended stochastic calculus for anticipating processes (see, for example, Nualart-Pardoux [6]), since the solutions to stochastic boundary problems are not in general adapted to the driving Brownian motion. In these papers, two types of problems have been considered. First to prove the existence and uniqueness of a solution for different kinds of equations, and secondly, to study the Markov properties of the solution.

Several kinds of Markov properties can be considered in connection with stochastic differential equations with boundary conditions. Throughout this paper, we will write

$$\mathcal{F}_1 \underset{\mathcal{F}_3}{\perp\!\!\!\perp} \mathcal{F}_2$$

to mean that the σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given $\mathcal{F}_3$.

Let $\{X_t, t \in T\}$ be a stochastic process indexed by a one-dimensional interval $T = [a,b] \subset I\!\!R$. If $S \subset T$, denote by $\mathcal{F}_S$ the σ-field generated by the random variables $\{X_t, t \in S\}$.

We say that $\{X_t, t \in T\}$ is a *Markov process* iff

$$\forall \quad t \in T \quad , \quad \mathcal{F}_{[a,t]} \underset{\mathcal{F}_{\{t\}}}{\perp\!\!\!\perp} \mathcal{F}_{[t,b]}.$$

We say that $\{X_t, t \in T\}$ is a *Markov field* iff

$$\forall \quad s \leq t, \quad s,t \in T, \quad \mathcal{F}_{[s,t]} \underset{\mathcal{F}_{\{s,t\}}}{\perp\!\!\!\perp} \mathcal{F}_{[a,b]-]s,t[}.$$

Finally, we say that $\{X_t, t \in T\}$ is a *germ Markov field* iff

$$\forall \quad s \leq t, \quad s,t \in T, \quad \mathcal{F}_{[s,t]} \underset{\mathcal{G}_{\{s,t\}}}{\perp\!\!\!\perp} \mathcal{F}_{[a,b]-]s,t[},$$

where $\mathcal{G}_{\{s,t\}} = \bigcap_{\varepsilon>0} \mathcal{F}_{]s-\varepsilon,s+\varepsilon[\cup]t-\varepsilon,t+\varepsilon[}$.

The above definitions can be adapted to the case of a finite parameter set. In general, solutions to stochastic differential equations with boundary conditions do not enjoy anyone of the properties listed above, except in some particular situations, like the linear case. For example, this type of negative results about the Markov property have been proved by Nualart and Pardoux [7,8] for first and second order equations, and by C. Donati–Martin [2] for first order equations with a linear diffusion coefficient. On the other hand, in [9], Ocone and Pardoux have proved that the Markov field property holds for almost all the quasi–linear equations (the general case is open).

A common feature in all these papers is the method used to obtain such negative results. This method can be described as follows. First a particular equation is considered, where the nonlinear term is replaced by zero or by a simple linear coefficient. For this equation, an explicit solution can be calculated, and the Markov property is shown by direct arguments. Then, an extended version of Girsanov theorem (cf. [4]) is used to obtain a new equivalent probability under which the law of this particular solution coincides with that of the original problem with a general nonlinear coefficient. Finally, the Markov property is translated into a factorization property of the Radon–Nikodym density. One of the main difficulties involved in this procedure is the computation of the Carleman-Fredholm determinant of a certain integral operator.

This method can be applied as well to discuss the Markov properties of stochastic difference equations (see Donati-Martin [1]).

In this paper we present another possible approach to the investigation of Markov properties for boundary value problems. The key point is the translation of the Markov property into the conditional independence of the white noise on two disjoint regions given the values of the process on the common boundary of these regions. This problem can be considered in an abstract form, and one can raise the following general question: Let Z_1 and Z_2 be two independent random variables; are Z_1 and Z_2 conditionally independent given some function $g(Z_1, Z_2)$? In Section 2 we present some sufficient conditions on the random variables Z_1 and Z_2 and on the function g for having an affirmative answer to this question. The main criterion in the absolutely continuous case is based on the computation of conditional densities by means of the co–area formula (see [3]). This application of the co–area formula has been inspired by the work of Ocone and Pardoux [9,10].

In Section 3, we apply the results obtained in Section 2 to study the Markov property of the second order stochastic difference equation with Dirichlet boundary conditions:

$$\left.\begin{array}{c}\Delta^2 X_n + f(X_{n+1}) = \xi_n \quad , \quad 0 \le n \le N-2 \\ X_0 = 0\,,\ X_N = 0\end{array}\right\}$$

where $\Delta^2 X_n = X_{n+2} - 2X_{n+1} + X_n$, and the random variables ξ_n are independent. This is the equation studied by C. Donati-Martin in [1], under the assumption that the variables ξ_n are standard Gaussian. With our method we are able to recover the main result in [1], which says that the solution is a Markov process if and only if the function f is affine, provided the laws of the variables ξ_n are absolutely continuous and with a strictly positive density. In the discrete case the Markov property always holds.

2. Sufficient conditions for conditional independence

In this section we will present some general criteria for the conditional independence of two independent random variables Z_1 and Z_2 when some function $g(Z_1, Z_2)$ is given. For an arbitrary function g we cannot expect this conditional independence to hold, but if g has the particular form exhibited in the proposition below, and if the random variables we are dealing with are discrete, then Z_1 and Z_2 are conditionally independent given $g(Z_1, Z_2)$.

Proposition 2.1. *Let $(M_1, \mathcal{M}_1)$, $(M_2, \mathcal{M}_2)$, $(A_1, \mathcal{A}_1)$, $(A_2, \mathcal{A}_2)$ be measurable spaces and let $g_1: M_2 \times A_1 \longrightarrow M_1$, $g_2: M_1 \times A_2 \longrightarrow M_2$ be two measurable functions such that the system of equations*

$$\left.\begin{aligned} x &= g_1(y, z_1) \\ y &= g_2(x, z_2) \end{aligned}\right\} \tag{2.1}$$

has a unique solution $(x, y) \in M_1 \times M_2$ for any given $(z_1, z_2) \in A_1 \times A_2$. Suppose that Z_1 and Z_2 are two independent random variables, defined in some probability space $(\Omega, \mathcal{F}, P)$ and taking values in A_1 and A_2 respectively. Consider the random variables X and Y defined by

$$\left.\begin{aligned} X(\omega) &= g_1(Y(\omega), Z_1(\omega)) \\ Y(\omega) &= g_2(X(\omega), Z_2(\omega)) \end{aligned}\right\} \tag{2.2}$$

Then, if the random variables X and Y are discrete it holds that

$$Z_1 \underset{X,Y}{\perp\!\!\!\perp} Z_2$$

PROOF: Fix sets $B_1 \in \mathcal{A}_1$, $B_2 \in \mathcal{A}_2$, and points $x \in M_1$, $y \in M_2$ such that $P\{X = x, Y = y\} > 0$. Using the uniqueness in the system (2.1), we have

$$\begin{aligned} P\{Z_1 \in B_1, Z_2 \in B_2 | X = x, Y = y\} \\ &= \frac{P\{Z_1 \in B_1, Z_2 \in B_2, X = x, Y = y\}}{P\{X = x, Y = y\}} \\ &= \frac{P\{Z_1 \in B_1, Z_2 \in B_2, x = g_1(y, Z_1), y = g_2(x, Z_2)\}}{P\{x = g_1(y, Z_1), y = g_2(x, Z_2)\}} \quad . \end{aligned}$$

By the independence of Z_1 and Z_2, this is equal to

$$\frac{P\{Z_1 \in B_1, x = g_1(y, Z_1)\}}{P\{x = g_1(y, Z_1)\}} \cdot \frac{P\{Z_2 \in B_2, y = g_2(x, Z_2)\}}{P\{y = g_2(x, Z_2)\}}.$$

Now if we take first $B_1 = A_1$ and then $B_2 = A_2$ in the preceding equality, we obtain that the above conditional probability is equal to the product

$$P\{Z_1 \in B_1 | X = x, Y = y\} P\{Z_2 \in B_2 | X = x, Y = y\} \quad ,$$

which completes the proof. □

Remark 1. In the above proposition it is sufficient to have the relation (2.1) for all (z_1, z_2) out of a set N of measure zero for the law of (Z_1, Z_2). On the other hand, if the variables Z_1 and Z_2 are discrete, then X and Y are also discrete and the proposition still holds.

Remark 2. If the system (2.1) has the particular form

$$\left.\begin{array}{ll} x & = g_1(z_1) \\ y & = g_2(x, z_2) \end{array}\right\}$$

then the requirement that the laws of the random variables X and Y are discrete is not necessary. Indeed, in this case we have

$$Z_1 \perp\!\!\!\perp Z_2 \Longrightarrow Z_1 \underset{X}{\perp\!\!\!\perp} Z_2 \Longrightarrow Z_1 \underset{X,Y}{\perp\!\!\!\perp} Z_2 \quad ,$$

since we can always enlarge the conditioning σ-field with events which belong to one of the σ–fields $\sigma(Z_1)$ or $\sigma(Z_2)$ (see Rozanov [11, page 57]).

If the random variables we are dealing with are not discrete, the conditional independence of Z_1 and Z_2 given X and Y is not true in general, as we will see in the next proposition. The following example illustrates this situation.

Example. Let A, B, C, D be independent random variables with the common law $N(0,1)$. Define

$$X = AY + B, \quad Y = CX + D.$$

Then the random vectors (A, B) and (C, D) are not conditionally independent given X, Y. Indeed, the four–dimensional vector (A, C, X, Y) has a density given by

$$f(a, c, x, y) = \psi(a)\psi(c)\psi(x - ay)\psi(y - cx)|1 - ac|,$$

where ψ is the standard normal density. The above density cannot be written in the form $\varphi_1(x, y, a)\varphi_2(x, y, c)$, and the conditional independence of A and C given X, Y is not true.

If the random variables X and Y are not discrete the result is quite different. The conditional independence does not hold unless the functions g_1 and g_2 verify some restrictive condition. In order to formulate this condition we introduce the following technical hypothesis on the system (2.1).

(H.1) Let A_1 and A_2 be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, with $n+m>2$. Consider C^1 functions $g_1\colon \mathbb{R}\times A_1 \longrightarrow \mathbb{R}$, $g_2\colon \mathbb{R}\times A_2 \longrightarrow \mathbb{R}$, such that the system

$$\left.\begin{aligned} x &= g_1(y,z_1) \\ y &= g_2(x,z_2) \end{aligned}\right\} \tag{2.3}$$

has a unique solution (x,y) for each $(z_1,z_2)\in V$, where V is an open subset of $A_1\times A_2$. We also assume that for all $(z_1,z_2)\in V$, and for x,y given by the system (2.3) we have

$$\text{(i)} \qquad \left|1-\frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial x}\right| \neq 0,$$

and

$$\text{(ii)} \qquad \|\nabla g_1\|\cdot\|\nabla g_2\| \neq 0,$$

where ∇g_1 and ∇g_2 denote the gradients of the functions g_1 and g_2 with respect to the variables z_1 and z_2, respectively.

Then we have:

Proposition 2.2. *Let g_1 and g_2 be functions satisfying the hypothesis* **(H.1)**. *Suppose that Z_1 and Z_2 are independent random vectors with absolutely continuous distributions such that $P\{(Z_1,Z_2)\in V\}=1$. Let X and Y be the random variables defined by*

$$\left.\begin{aligned} X(\omega) &= g_1(Y(\omega),Z_1(\omega)) \\ Y(\omega) &= g_2(X(\omega),Z_2(\omega)) \end{aligned}\right\} \tag{2.4}$$

Then,

$$Z_1 \underset{X,Y}{\perp\!\!\!\perp} Z_2$$

if and only if there exist measurable functions $F_1:\mathbb{R}^2\times A_1\to\mathbb{R}$, $F_2:\mathbb{R}^2\times A_2\to\mathbb{R}$, such that

$$\left|1-\frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial x}\right|(X,Y,Z_1,Z_2) = F_1(X,Y,Z_1)F_2(X,Y,Z_2), \quad \text{a.s.} \tag{2.5}$$

PROOF: Let $\psi : V \longrightarrow \mathbb{R}^2$ be the function which maps $(z_1, z_2) \in V$ into the solution (x, y) of the system (2.3). Because of the Implicit Function Theorem and condition (i) ψ is of class $\mathcal{C}^1$.

Set $\delta = \left[1 - \frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial x}\right]$, $\theta_1 = \| \nabla g_1 \|$, and $\theta_2 = \| \nabla g_2 \|$. Using the formulas

$$\frac{\partial \psi_1}{\partial z_1^i} = \delta^{-1}\frac{\partial g_1}{\partial z_1^i} \quad , \quad \frac{\partial \psi_1}{\partial z_2^j} = \delta^{-1}\frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial z_2^j} \quad ,$$
$$\frac{\partial \psi_2}{\partial z_1^i} = \delta^{-1}\frac{\partial g_2}{\partial x}\frac{\partial g_1}{\partial z_1^i} \quad , \quad \frac{\partial \psi_2}{\partial z_2^j} = \delta^{-1}\frac{\partial g_2}{\partial z_2^j} \quad ,$$

we get that the generalized Jacobian $J\psi = \left[\det\left(< \nabla\psi_i, \nabla\psi_j >\right)_{1\le i,j\le 2}\right]^{1/2}$ is equal to $|\delta|^{-1}\theta_1\theta_2$.

Fix two Borel sets $B_1 \subset A_1$ and $B_2 \subset A_2$, such that $B_1 \times B_2 \subset V$. We will denote by $\mathcal{H}^m$ the Hausdorff measure of dimension m. Using the co-area formula (see [3, 10]) we can obtain the following expression for the conditional probability of $\{Z_1 \in B_1, Z_2 \in B_2\}$ given $X = x, Y = y$:

$$(2.6) \quad P\{Z_1 \in B_1, Z_2 \in B_2 | \psi(Z_1, Z_2) = (x, y)\} = [f_{X,Y}(x, y)]^{-1} \times \int_{\psi^{-1}(x,y)} \mathbf{1}_{B_1\times B_2}(z_1, z_2)|\delta|\theta_1^{-1}\theta_2^{-1} f_{Z_1}(z_1) f_{Z_2}(z_2)\, d\mathcal{H}^{n+m-2}(z_1, z_2),$$

for almost all (x, y) with respect to $P_{X,Y}$ (law of (X, Y)), and where f_{Z_1}, f_{Z_2} and $f_{X,Y}$ denote the densities of the random vectors Z_1, Z_2 and (X, Y), respectively.

Observe that from (2.3) we deduce $\psi^{-1}(x, y) = [R_1(x, y) \times R_2(x, y)] \cap V$, where

$$R_1(x, y) = \{z_1 \in A_1 \, : \, x = g_1(y, z_1)\},$$
$$R_2(x, y) = \{z_2 \in A_2 \, : \, x = g_2(x, z_2)\}.$$

Therefore, we can write

$$(2.7) \quad P\{Z_1 \in B_1, Z_2 \in B_2 | \psi(Z_1, Z_2) = (x, y)\} = [f_{X,Y}(x, y)]^{-1} \int_{R_1(x,y)\times R_2(x,y)} \mathbf{1}_{B_1\times B_2}(z_1, z_2) \times |\delta|\theta_1^{-1}\theta_2^{-1} f_{Z_1}(z_1) f_{Z_2}(z_2)\, d\mathcal{H}^{n+m-2}(z_1, z_2).$$

We claim that

$$(2.8) \quad \int_{\mathbb{R}^2} \left[\int_{R_1(x,y)\times R_2(x,y)} \left(\mathbf{1}_{V^c}|\delta|\theta_1^{-1}\theta_2^{-1} f_{Z_1} f_{Z_2}\right)(x, y, z_1, z_2) \times d\mathcal{H}^{n+m-2}(z_1, z_2)\right] dxdy = 0,$$

with the convention $0 \cdot \infty = 0$. In fact, fix $(x, y) \in I\!R^2$. On the set $\{(z_1, z_2) \in A_1 \times A_2 : \delta(x, y, z_1, z_2) = 0\}$ clearly the integrand in the expression (2.8) vanishes. So it suffices to consider the integral on the set $V^c \cap \{\delta \neq 0\}$, and it is enough to show that for any fixed point $(x^0, y^0, z_1^0, z_2^0) \in I\!R^2 \times A_1 \times A_2$ which verifies (2.3) and such that $\delta(x^0, y^0, z_1^0, z_2^0) \neq 0$, there exists a neighbourhood U of (x^0, y^0, z_1^0, z_2^0) such that

$$\int_{I\!R^2} \Big[\int_{R_1(x,y) \times R_2(x,y)} \left(\mathbf{1}_{V^c \cap U(x,y)} |\delta| \theta_1^{-1} \theta_2^{-1} f_{Z_1} f_{Z_2} \right)(x, y, z_1, z_2) \times d\mathcal{H}^{n+m-2}(z_1, z_2) \Big] dx dy = 0, \tag{2.9}$$

where $U(x, y) = \{(z_1, z_2) \in A_1 \times A_2 : (x, y, z_1, z_2) \in U\}$. By the Implicit Function Theorem, we can choose the neighbourhood U in such a way that there exist neighbourhoods U_1 and U_2 of (x^0, y^0) and (z_1^0, z_2^0), respectively, and a continuously differentiable function $\varphi : U_2 \to U_1$, such that $U = U_1 \times U_2$, and for any $(x, y, z_1, z_2) \in U$ we have

$$\left.\begin{aligned} x &= g_1(y, z_1) \\ y &= g_2(x, z_2) \end{aligned}\right\} \quad \text{if and only if} \quad (x, y) = \varphi(z_1, z_2).$$

Now we can apply the co–area formula to the function φ and we obtain that the left hand side of (2.9) is equal to

$$\int_{U_2} \mathbf{1}_{V^c}(z_1, z_2) f_{Z_1}(z_1) f_{Z_2}(z_2) dz_1 \, dz_2 = 0,$$

and (2.8) holds. As a consequence, the equality (2.7) holds for any rectangle $B_1 \times B_2 \subset A_1 \times A_2$, not necessarily included in V.

Let us now turn to the proof of the proposition. Suppose first that the factorization condition (2.5) holds. Then the equality (2.7) becomes

$$\begin{aligned} P\{Z_1 \in B_1, Z_2 \in B_2 | \psi(Z_1, Z_2) = (x, y)\} &= [f_{X,Y}(x, y)]^{-1} \\ &\times \left(\int_{R_1(x,y)} \mathbf{1}_{B_1}(z_1) F_1(x, y, z_1) \theta_1^{-1}(x, y, z_1) f_{Z_1}(z_1) \mathcal{H}^{n-1}(dz_1) \right) \\ &\times \left(\int_{R_2(x,y)} \mathbf{1}_{B_2}(z_2) F_2(x, y, z_2) \theta_2^{-1}(x, y, z_2) f_{Z_2}(z_2) \mathcal{H}^{m-1}(dz_2) \right) \end{aligned} \tag{2.10}$$

for $P_{X,Y}$-almost all (x, y). This factorization of the conditional probability implies the conditional independence of Z_1 and Z_2 given X and Y.

Conversely, suppose Z_1 and Z_2 are conditionally independent given X and Y. Fix two Borel subsets $B_1 \subset A_1$, and $B_2 \subset A_2$. Using the co-area formula we arrive, in the same way as before, to

$$P\{Z_1 \in B_1 | \psi(Z_1, Z_2) = (x, y)\} = [f_{X,Y}(x, y)]^{-1}$$
$$\times \int_{\psi^{-1}(x,y)} \mathbf{1}_{B_1}(z_1) |\delta(x, y, z_1, z_2)| \theta_1^{-1}(x, y, z_1) \theta_2^{-1}(x, y, z_2)$$
$$\times f_{Z_1}(z_1) f_{Z_2}(z_2)\, d\mathcal{H}^{n+m-2}(z_1, z_2) = [f_{X,Y}(x, y)]^{-1}$$
$$\times \int_{R_1(x,y)} \mathbf{1}_{B_1}(z_1) \theta_1^{-1}(x, y, z_1) f_{Z_1}(z_1) \Lambda_1(x, y, z_1)\, \mathcal{H}^{n-1}(dz_1),$$

where $\Lambda_1(x, y, z_1) = \int_{R_2(x,y)} \theta_2^{-1}(x, y, z_2) f_{Z_2}(z_2) |\delta(x, y, z_1, z_2)|\, \mathcal{H}^{m-1}(dz_2)$, and

$$P\{Z_2 \in B_2 | \psi(Z_1, Z_2) = (x, y)\} = [f_{X,Y}(x, y)]^{-1}$$
$$\times \int_{\psi^{-1}(x,y)} \mathbf{1}_{B_2}(z_2) |\delta(x, y, z_1, z_2)| \theta_1^{-1}(x, y, z_1) \theta_2^{-1}(x, y, z_2)$$
$$\times f_{Z_1}(z_1) f_{Z_2}(z_2)\, d\mathcal{H}^{n+m-2}(z_1, z_2) = [f_{X,Y}(x, y)]^{-1}$$
$$\times \int_{R_2(x,y)} \mathbf{1}_{B_2}(z_2) \theta_2^{-1}(x, y, z_2) f_{Z_2}(z_2) \Lambda_2(x, y, z_2)\, \mathcal{H}^{m-1}(dz_2),$$

where $\Lambda_2(x, y, z_2) = \int_{R_1(x,y)} \theta_1^{-1}(x, y, z_1) f_{Z_1}(z_1) |\delta(x, y, z_1, z_2)|\, \mathcal{H}^{n-1}(dz_1)$.

The product of these two expressions must agree with (2.7) almost surely with respect to $P_{X,Y}$. That means,

$$[f_{X,Y}(x, y)]^{-2} \int_{R_1(x,y) \times R_2(x,y)} \mathbf{1}_{B_1}(z_1) \mathbf{1}_{B_2}(z_2) \theta_1^{-1}(x, y, z_1) \theta_2^{-1}(x, y, z_2)$$
$$\times f_{Z_1}(z_1) f_{Z_2}(z_2) \Lambda_1(x, y, z_1) \Lambda_2(x, y, z_2)\, \mathcal{H}^{n-1}(dz_1) \mathcal{H}^{m-1}(dz_2) =$$
$$[f_{X,Y}(x, y)]^{-1} \int_{R_1(x,y) \times R_2(x,y)} \mathbf{1}_{B_1}(z_1) \mathbf{1}_{B_2}(z_2) \theta_1^{-1}(x, y, z_1) \theta_2^{-1}(x, y, z_2)$$
$$\times f_{Z_1}(z_1) f_{Z_2}(z_2) |\delta(x, y, z_1, z_2)|\, \mathcal{H}^{n-1}(dz_1) \mathcal{H}^{m-1}(dz_2).$$

Therefore, $f_{X,Y}(x, y)^{-1} \Lambda_1(x, y, z_1) \Lambda_2(x, y, z_2)$ and $|\delta(x, y, z_1, z_2)|$ must coincide on $\psi^{-1}(x, y)$, a.e. with respect to the measure

$$f_{Z_1}(z_1) f_{Z_2}(z_2) \cdot [\mathcal{H}^{n-1} \otimes \mathcal{H}^{m-1}],$$

and in consequence with respect to the conditional law of (Z_1, Z_2) given $(X, Y) = (x, y)$. This happens with probability 1 with respect to $P_{X,Y}$, and the conclusion (2.5) follows. □

Notice that the variables x, y in the above proposition can also be multidimensional (with the same dimension). In that case, the absolute value in the factorization condition (2.5) has to be replaced by the absolute value of the determinant of identity minus the product of two Jacobian matrices.

In order to apply this result we need to characterize the functions g_1 and g_2 for which the factorization (2.5) holds. This is the objective of the next lemma.

Lemma 2.3. *Let G_1 and G_2 be continuously differentiable functions defined in open subsets $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$, respectively. Let V be an open subset of $V_1 \times V_2$ such that $V \subset \{\| \nabla G_1 \| + \| \nabla G_2 \| \neq 0\}$. The following two statements are equivalent:*

(1) $|1 - G_1(z_1)G_2(z_2)| = F_1(z_1)F_2(z_2)$ for all $(z_1, z_2) \in V$ and for some measurable functions F_1 and F_2.

(2) We have

$$\frac{\partial G_1}{\partial z_1^i}(z_1)\frac{\partial G_2}{\partial z_2^j}(z_2) = 0, \quad \text{for all} \quad i,j \quad \text{and for all} \quad (z_1, z_2) \in V.$$

PROOF: (1) $\Rightarrow$ (2):

Suppose we have $\frac{\partial G_1}{\partial z_1^i}(z_1) \neq 0$ and $\frac{\partial G_2}{\partial z_2^j}(z_2) \neq 0$, for some fixed $1 \leq i \leq n$, $1 \leq j \leq m$ and $(z_1, z_2) \in V$. This implies that we can choose a small open rectangle $U \subset V$, such that in U the above partial derivatives do not vanish and in addition we have

$$G_1(z_1) \neq 0 \quad , \quad G_2(z_2) \neq 0 \quad \text{and} \quad 1 - G_1(z_1)G_2(z_2) \neq 0.$$

Differentiating with respect to z_1^i the expression in (1) in the set U (notice that from the equation in (1) we deduce that F_1 and F_2 are differentiable in U), we obtain

$$0 \neq \frac{\pm\partial G_1}{\partial z_1^i} G_2 = \frac{\partial F_1}{\partial z_1^i} F_2 \quad ,$$

which implies that $\frac{F_2}{G_2}$ is in fact a constant C, since we can write

$$\frac{F_2}{G_2} = \frac{\pm\partial G_1}{\partial z_1^i} \left(\frac{\partial F_1}{\partial z_1^i}\right)^{-1} .$$

That means we have

$$1 - G_1G_2 = CF_1G_2 \ \Rightarrow \frac{1}{G_2} = G_1 + CF_1,$$

and G_2 cannot depend on z_2 which is in contradiction with the fact that its partial derivative with respect to z_2^j does not vanish in U.

(2) $\Rightarrow$ (1) : Consider the open subsets of V defined by

$$U_1 = \{(z_1, z_2) \in V, \ \frac{\partial G_1}{\partial z_1^i}(z_1) \neq 0 \text{ for some } i\},$$

$$U_2 = \{(z_1, z_2) \in V, \ \frac{\partial G_2}{\partial z_2^j}(z_2) \neq 0 \text{ for some } j\}.$$

Condition (2) implies that U_1 and U_2 are disjoint and their union is the whole set V because we have $V \subset \{\| \nabla G_1 \| + \| \nabla G_2 \| \neq 0\}$. Consequently, if we define

$$F_1(z_1) = 1 - G_1(z_1)G_2(z_2)\mathbf{1}_{U_1}, \quad F_2(z_2) = 1 - G_1(z_1)G_2(z_2)\mathbf{1}_{U_2},$$

then the equality (1) will be true on V. □

Notice that for the implication (1) ⇒ (2) we do not need the condition $V \subset \{\| \nabla G_1 \| + \| \nabla G_2 \| \neq 0\}$.

If we apply Lemma 2.3 to the functions $G_1 = \dfrac{\partial g_1}{\partial y}$ and $G_2 = \dfrac{\partial g_2}{\partial x}$, where g_1 and g_2 satisfy hypothesis (H.1) we obtain the following result.

Lemma 2.4. *Let g_1 and g_2 be two functions satisfying hypothesis* **(H.1)**. *Assume moreover that the density of (Z_1, Z_2) verifies $f_{Z_1,Z_2} > 0$ a.e. on V. Then condition (2.5) implies that for all $1 \leq i, k \leq n, k \neq i$ and $1 \leq j, l \leq m, l \neq j$ we have*

$$\frac{\partial}{\partial y}\left(\frac{\partial g_1/\partial z_1^k}{\partial g_1/\partial z_1^i}\right) \cdot \frac{\partial}{\partial x}\left(\frac{\partial g_2/\partial z_2^l}{\partial g_2/\partial z_2^j}\right) = 0, \tag{2.11}$$

on the set

$$\left\{\frac{\partial g_1}{\partial z_1^i} \neq 0 \quad \text{and} \quad \frac{\partial g_2}{\partial z_2^j} \neq 0\right\} \bigcap \{(x, y, z_1, z_2) \in \mathbb{R} \times V : x = g_1(y, z_1), \, y = g_2(x, z_2)\}.$$

Proof: Fix $\xi^0 = (x^0, y^0, z_1^0, z_2^0)$ in the above set. The conditions $\dfrac{\partial g_1}{\partial z_1^i}(\xi^0) \neq 0$ and $\dfrac{\partial g_2}{\partial z_2^j}(\xi^0) \neq 0$ allow to apply the Implicit Function Theorem and to write locally the system (2.3) in the form

$$z_1^i = h_1(x, y, \hat{z}_1^i) \quad , z_2^j = h_2(x, y, \hat{z}_2^j) \quad ,$$

where

$$\hat{z}_1^i = (z_1^1, \ldots, z_1^{i-1}, z_1^{i+1}, \ldots, z_1^n),$$

and

$$\hat{z}_2^j = (z_2^1, \ldots, z_2^{j-1}, z_2^{j+1}, \ldots, z_2^m).$$

That means there exists neighbourhoods U_1 of (x^0, y^0), V_1 of $z_1^{0,i}$, $\hat{V}_1$ of $\hat{z}_1^{0,i}$, V_2 of $z_2^{0,j}$, $\hat{V}_2$ of $\hat{z}_2^{0,j}$, and functions

$$h_1 : U_1 \times \hat{V}_1 \to V_1, \quad h_2 : U_1 \times \hat{V}_2 \to V_2,$$

such that for any $(x, y, z_1, z_2) \in U_1 \times V_1 \times \hat{V}_1 \times V_2 \times \hat{V}_2$

$$\left.\begin{array}{ll} x & = g_1(y, z_1) \\ y & = g_2(x, z_2) \end{array}\right\} \text{ if and only if } \left\{\begin{array}{lll} z_1^i & = & h_1(x, y, \hat{z}_1^i) \\ z_2^j & = & h_2(x, y, \hat{z}_2^j) \end{array}\right. .$$

Set $\bar{U}_1 = U_1 \times \hat{V}_1 \times \hat{V}_2$, and

$$G_1(x, y, \hat{z}_1^i) = \frac{\partial g_1}{\partial y}(y, z_1)|_{z_1^i = h_1(x, y, \hat{z}_1^i)}$$

and

$$G_2(x, y, \hat{z}_2^j) = \frac{\partial g_2}{\partial x}(y, z_2)|_{z_2^j = h_2(x, y, \hat{z}_2^j)}.$$

Condition (2.5) says that for all $(x, y, \hat{z}_1^i, \hat{z}_2^j) \in \bar{U}_1$, almost surely with respect to the law of $(X, Y, \hat{Z}_1^i, \hat{Z}_2^j)$, we have

$$\begin{aligned} (2.12) \quad & \left|1 - G_1(x, y, \hat{z}_1^i) G_2(x, y, \hat{z}_2^j)\right| \\ & = F_1(x, y, z_1)|_{z_1^i = h_1(x, y, \hat{z}_1^i)} F_2(x, y, z_2)|_{z_2^j = h_2(x, y, \hat{z}_2^j)}. \end{aligned}$$

We claim that on the set $\bar{U}_1$ the law of $W := (X, Y, \hat{Z}_1^i, \hat{Z}_2^j)$ is equivalent to the Lebesgue measure. In fact the density of W on $\bar{U}_1$ is given by

$$\left[f_{Z_1} f_{Z_2} |\delta| \left| \frac{\partial g_1}{\partial z_1^i} \right|^{-1} \left| \frac{\partial g_2}{\partial z_2^j} \right|^{-1} \right] ((h_1(x, y, \hat{z}_1^i), h_2(x, y, \hat{z}_2^j), \hat{z}_1^i, \hat{z}_2^j)).$$

Then using the change of variable formula and the fact that $f_{Z_1} f_{Z_2}$ is strictly positive almost everywhere with respect to the Lebesgue measure, we deduce that the Lebesgue measure of the set $\{(x, y, \hat{z}_1^i, \hat{z}_2^j) \in \bar{U}_1 : f_{Z_1} f_{Z_2}(x, y, \hat{z}_1^i, \hat{z}_2^j) = 0\}$ is zero.

As a consequence we can assume that the equality (2.12) holds for all $(x, y, \hat{z}_1^i, \hat{z}_2^j) \in \bar{U}_1$. In fact, we can find two points $\hat{\zeta}_1 \in \hat{V}_1$, and $\hat{\zeta}_2 \in \hat{V}_2$ such that the equality (2.12) is satisfied for $\hat{z}_1^i = \hat{\zeta}_1$, $\hat{z}_2^j = \hat{\zeta}_2$, and for almost all $(x, y) \in U_1$, and on the other hand, if we fix $\hat{z}_1^i = \hat{\zeta}_1$ or $\hat{z}_2^j = \hat{\zeta}_2$, then the equality holds true for almost all $(x, y, \hat{z}_2^j)$ and for almost all $(x, y, \hat{z}_1^i)$, respectively. Define

$$\hat{F}_1(x, y, \hat{z}_1^i) = F_1(x, y, z_1)|_{z_1^i = h_1(x, y, \hat{z}_1^i)},$$

and

$$\hat{F}_2(x,y,\hat{z}_2^j) = F_2(x,y,z_2)_{|_{z_2^j=h_2(x,y,\hat{z}_2^j)}}.$$

Then is is not difficult to see that the equality (2.12) remains true a.s. if we replace the functions $\hat{F}_1$ and $\hat{F}_2$ by

$$\hat{F}_1^0(x,y,\hat{z}_1^i) = \left|1 - G_1(x,y,\hat{z}_1^i)G_2(x,y,\zeta_2)\right|$$

and

$$\hat{F}_2^0(x,y,\hat{z}_2^j) = \left|\frac{1 - G_1(x,y,\zeta_1)G_2(x,y,\hat{z}_2^j)}{1 - G_1(x,y,\zeta_1)G_2(x,y,\zeta_2)}\right|.$$

By the continuity of the above functions, the equality (2.12) is satisfied everywhere.

Then we can apply Lemma 2.3 to the functions G_1 and G_2, for each fixed (x,y), and to the variables $\hat{z}_1^i$ and $\hat{z}_2^j$, varying in $\hat{V}_1$ and $\hat{V}_2$, respectively. So we obtain

$$\frac{\partial G_1}{\partial z_1^k}(x,y,\hat{z}_1^i)\frac{\partial G_2}{\partial z_2^l}(x,y,\hat{z}_2^j) = 0,$$

for all $k \neq i$, $l \neq j$. Therefore, it suffices to compute these derivatives and to compare their values with the factors appearing in the left hand side of (2.11). That means we claim that

$$\frac{\partial G_1}{\partial z_1^k}(x,y,\hat{z}_1^i) = \left(\frac{\partial g_1}{\partial z_1^i}\right)\cdot\frac{\partial}{\partial y}\left(\frac{\partial g_1/\partial z_1^k}{\partial g_1/\partial z_1^i}\right)_{|_{z_1^i=h_1(x,y,\hat{z}_1^i)}}, \tag{2.13}$$

and

$$\frac{\partial G_2}{\partial z_2^l}(x,y,\hat{z}_2^j) = \left(\frac{\partial g_2}{\partial z_2^j}\right)\cdot\frac{\partial}{\partial x}\left(\frac{\partial g_2/\partial z_2^l}{\partial g_2/\partial z_2^j}\right)_{|_{z_2^j=h_2(x,y,\hat{z}_2^j)}}. \tag{2.14}$$

In fact, using the equality $\dfrac{\partial h_1}{\partial z_1^k} = -\dfrac{\partial g_1/\partial z_1^k}{\partial g_1/\partial z_1^i}$, we get

$$\begin{aligned}\frac{\partial G_1}{\partial z_1^k}(x,y,\hat{z}_1^i) &= \frac{\partial}{\partial z_1^k}\left(\frac{\partial g_1}{\partial y}(y,z_1)_{|_{z_1^i=h_1(x,y,\hat{z}_1^i)}}\right) = \\ &= \left[\frac{\partial^2 g_1}{\partial z_1^i \partial y}(y,z_1)\frac{-\frac{\partial g_1}{\partial z_1^k}(y,z_1)}{\frac{\partial g_1}{\partial z_1^i}(y,z_1)} + \frac{\partial^2 g_1}{\partial z_1^k \partial y}(y,z_1)\right]_{z_1^i=h_1(x,y,\hat{z}_1^i)}.\end{aligned} \tag{2.15}$$

The right hand side of (2.15) coincides with

$$\left(\frac{\partial g_1}{\partial z_1^i}\right)^{-1}\left(-\frac{\partial}{\partial y}\left(\frac{\partial g_1}{\partial z_1^i}\right)\frac{\partial g_1}{\partial z_1^k}+\frac{\partial}{\partial y}\left(\frac{\partial g_1}{\partial z_1^k}\right)\frac{\partial g_1}{\partial z_1^i}\right) = \left(\frac{\partial g_1}{\partial z_1^i}\right)\cdot\frac{\partial}{\partial y}\left(\frac{\partial g_1/\partial z_1^k}{\partial g_1/\partial z_1^i}\right),$$

and this implies (2.13). The proof of (2.14) is analogous. □

3. Application to a second order stochastic difference equation with boundary conditions

In this section we will make use of the preceding results to study the Markov property of the solution to the one-dimensional second order difference equation

$$\Delta^2 X_n + f(X_{n+1}) = \xi_n \quad , \quad 0 \le n \le N-2,$$

with Dirichlet boundary conditions $X_0 = 0$, $X_N = 0$.

Here Δ^2 is the second order difference operator $\Delta^2 X_n = \Delta(\Delta X_n) = X_{n+2} - 2X_{n+1} + X_n$, f is a real function and $\{\xi_n,\, 0 \le n \le N-2\}$ is a given "noise" process.

In [1], C. Donati-Martin has studied this equation in the case where $\{\xi_n\,,\, 0 \le n \le N-2\}$ is a sequence of independent $N(0,1)$ random variables. Using the method of change of measures, she proved that if the process $\{(X_n, \Delta X_n)\,,\, 0 \le n \le N-1\}$ is a Markov process (or even only a Markov field) and f is of class $\mathcal{C}^2$, then f must be affine, and conversely, if f is affine, the solution is a Markov process.

We will prove the equivalence

$$\big\{(X_n, \Delta X_n)\,,\, 0 \le n \le N-1\big\} \quad \text{is a Markov process} \quad \Leftrightarrow \quad f \quad \text{is affine}$$

for absolutely continuous variables ξ_n whose support is the whole real line, and that $\big\{(X_n, \Delta X_n)\,,\, 0 \le n \le N-1\big\}$ is always a Markov process if they are discrete.

We first recall the existence and uniqueness theorem for the above equation given by Donati-Martin, which is a deterministic result and does not depend on the law of $\{\xi_n\}_n$. Let $\{\xi_n\,,\, 0 \le n \le N-2\}$ be a sequence real numbers, and consider the following system of $N+1$ equations on the unknowns $\{X_n\ ,\ 0 \le n \le N\}$:

$$\left.\begin{aligned} \Delta^2 X_n + f(X_{n+1}) = \xi_n \quad &, \quad 0 \le n \le N-2 \\ X_0 = 0\,,\, X_N = 0 \end{aligned}\right\} \tag{3.1}$$

We have the following existence and uniqueness result.

Theorem 3.1. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is non-increasing, then (3.1) has a unique solution $\{X_n\,,\, 0 \leq n \leq N\}$.*

PROOF: Denote by A the matrix

$$A = \begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

which is negative-definite, as it can be easily seen. Suppose that $X^1 = (X^1_1, \ldots, X^1_{N-1})$ and $X^2 = (X^2_1, \ldots, X^2_{N-1})$ are two solutions of (3.1). Write $f(X^i) = \big(f(X^i_1), \ldots, f(X^i_{N-1})\big)$, $i = 1, 2$, and $\xi = (\xi_0, \ldots, \xi_{N-2})$. Then the system (3.1) can be written in matricial form as

$$AX^i + f(X^i) = \xi \quad , \quad i = 1, 2 \quad .$$

Thus,

$$A(X^1 - X^2) + f(X^1) - f(X^2) = 0,$$

and, therefore, taking scalar products with $X^1 - X^2$,

$$\langle A(X^1 - X^2), X^1 - X^2\rangle + \langle f(X^1) - f(X^2), X^1 - X^2\rangle = 0.$$

But f is non-increasing and A is negative-definite, so that both summands must be nonpositive, and consequently equal to zero. Since the first one can only be zero when $X^1 = X^2$, we arrive to this conclusion.

To show the existence, fix a vector $\xi \in \mathbb{R}^{N-1}$ and define

$$\begin{aligned} \psi_\xi : \mathbb{R}^{N-1} &\longrightarrow \mathbb{R}^{N-1} \\ X &\longrightarrow \xi - (A + f)(X) \end{aligned}$$

We want to see that there exists a point $X_\xi \in \mathbb{R}^{N-1}$ such that $\psi_\xi(X_\xi) = 0$. Using that $-A$ is positive-definite and Schwarz inequality,

$$\begin{aligned} \langle \psi_\xi(X), X\rangle &= \langle \xi, X\rangle - \langle AX, X\rangle - \langle f(X), X\rangle \\ &= \langle -AX, X\rangle + \langle \xi - f(0), X\rangle + \langle f(0) - f(X), X\rangle \\ &\geq \langle -AX, X\rangle + \langle \xi - f(0), X\rangle \\ &\geq \lambda \cdot \|X\|^2 + \langle \xi - f(0), X\rangle \\ &\geq \lambda \cdot \|X\|^2 - \|\xi - f(0)\| \cdot \|X\| \xrightarrow[\|X\| \to +\infty]{} +\infty \end{aligned}$$

for some $\lambda > 0$. Thus, we deduce the existence of $\rho > 0$ such that

$$\forall X, ||X|| = \rho \Rightarrow \langle \psi_\xi(X), X \rangle \geq 0$$

and, in this situation, Lemma 4.3. in Lions [5, page 53] applies and gives us the existence of X_ξ verifying $\psi_\xi(X_\xi) = 0$. □

Now let $\{\xi_n, 0 \leq n \leq N-2\}$ be a sequence of independent random variables, and consider the sequence of random variables $\{X_n, 0 \leq n \leq N\}$ defined by the system (3.1). We want to investigate when the 2-dimensional process $\{(X_n, \Delta X_n), 0 \leq n \leq N-1\}$ is a Markov process. Actually we will consider the process $\{(X_n, X_{n+1}), 0 \leq n \leq N-1\}$ which generates the same σ-fields than the previous one. The Markov property for this two-dimensional process means that for every p, $0 \leq p \leq N-1$,

$$\{X_n, 0 \leq n \leq p+1\} \underset{X_p, X_{p+1}}{\perp\!\!\!\perp} \{X_n, p \leq n \leq N\} \tag{3.2}$$

Notice that for $p = 0, 1, N-2, N-1$, the conditional independence (3.2) is obvious. Therefore, we will assume that p is such that $2 \leq p \leq N-3$. In order to apply Propositions 2.1 and 2.2 we will show first the following properties.

(1) It holds the equivalence

$$\{X_n, 0 \leq n \leq p+1\} \underset{X_p, X_{p+1}}{\perp\!\!\!\perp} \{X_n, p \leq n \leq N\} \iff$$
$$\{\xi_n, 0 \leq n \leq p-1\} \underset{X_p, X_{p+1}}{\perp\!\!\!\perp} \{\xi_n, p \leq n \leq N-2\}$$

(2) There exist functions g_1 and g_2 such that

$$\left.\begin{array}{rcl} X_p &=& g_1(X_{p+1}, \xi_0, \ldots, \xi_{p-1}) \\ X_{p+1} &=& g_2(X_p, \xi_p, \ldots, \xi_{N-2}) \end{array}\right\} \tag{3.3}$$

and this system has a unique solution (X_p, X_{p+1}) for any $(\xi_0, \ldots, \xi_{N-2}) \in \mathbb{R}^{N-1}$.

Lemma 3.2. *Let $\{\xi_n, 0 \leq n \leq N-2\}$ be a sequence of independent random variables, and suppose that f is non-increasing. Let $\{X_n, 0 \leq n \leq N\}$ be the solution of (3.1). Then the above properties (1) and (2) are true for all p, $2 \leq p \leq N-3$. Moreover if f is of class C^r, with $r \geq 1$, then g_1 and g_2 are also of class C^r.*

Proof: Fix p, $2 \leq p \leq N-3$. Property (1) is immediate. Indeed, from the system (3.1) it is clear that the random variables $\{X_n, 0 \leq n \leq p+1\}$ are measurable with respect to the σ-field generated by $\{\xi_n, 0 \leq n \leq p-1\}$ and by X_p, X_{p+1}, and similarly, the random variables $\{X_n, p \leq n \leq N\}$

are measurable with respect to the σ–field generated by $\{\xi_n, p \le n \le N-2\}$ and by X_p, X_{p+1}. Consequently, by the elementary properties of the conditional independence it follows that the implication $\Longleftarrow$ in (1) holds. The converse implication is proved by the same argument.

Let us turn to the proof of Property (2). Fix p, $2 \le p \le N-3$, and consider the system of equations

$$\left.\begin{array}{l} \Delta^2 X_n + f(X_{n+1}) = \xi_n \quad , \quad 0 \le n \le p-1 \\ X_0 = 0 \quad , \quad X_{p+1} \text{ given} \end{array}\right\}$$

The equivalent system in the unknowns $X_1, \dots, X_p$ is

$$\text{(3.4)} \qquad \left.\begin{array}{rcl} -2X_1 + X_2 + f(X_1) & = & \xi_0 \\ X_1 - 2X_2 + X_3 + f(X_2) & = & \xi_1 \\ \vdots & & \vdots \\ X_{p-2} - 2X_{p-1} + X_p + f(X_{p-1}) & = & \xi_{p-2} \\ X_{p-1} - 2X_p + f(X_p) & = & \xi_{p-1} - X_{p+1} \end{array}\right\}$$

and it can be treated exactly as in the proof of Theorem 3.1. That means this system of equations has a unique solution, and this implies that X_p is a function of $(X_{p+1}, \xi_0, \dots, \xi_{p-1})$.

On the other hand, from the existence of a unique solution for the system

$$\text{(3.5)} \qquad \left.\begin{array}{rcl} -2X_{p+1} + X_{p+2} + f(X_{p+1}) & = & \xi_p - X_p \\ X_{p+1} - 2X_{p+2} + X_{p+3} + f(X_{p+2}) & = & \xi_{p+1} \\ \vdots & & \vdots \\ -2X_{N-1} + X_{N-2} + f(X_{N-1}) & = & \xi_{N-2} \end{array}\right\}$$

it follows that X_{p+1} is a function of $X_p, \xi_p, \dots, \xi_{N-2}$. Moreover, putting together both systems we obtain (3.1), and this ensures the uniqueness of (X_p, X_{p+1}).

Finally, it is clear from the systems (3.4) and (3.5) and by the Implicit Function Theorem that g_1 and g_2 have the same smoothness properties than f. This completes the proof of the lemma. □

Using this lemma we can now state the following result about the Markov property.

Theorem 3.3. *Suppose the variables $\{\xi_n, 0 \le n \le N-2\}$ are independent and have discrete laws. Let $\{X_n, 0 \le n \le N\}$ be the solution to (3.1), with f nonincreasing. Then, $\{(X_n, \Delta X_n), 0 \le n \le N-1\}$ is a Markov process.*

Proof: In view of (1) of Lemma 3.2, we only need to apply Proposition 2.1 for each fixed p, $2 \le p \le N-3$, and $(Z_1^1, \dots, Z_1^n) = (\xi_0, \dots, \xi_{p-1})$,

$(Z_2^1, \ldots, Z_2^m) = (\xi_p, \ldots, \xi_{N-2})$, $X = X_p$, $Y = X_{p+1}$. The random variables X and Y are obviously discrete and the conclusion follows immediatly. □

Theorem 3.4. *Suppose the variables $\{\xi_n\ ,\ 0 \le n \le N-2\}$ are independent and have absolutely continuous distributions. Let $\{X_n\ ,\ 0 \le n \le N\}$ be the solution to (3.1) with f nonincreasing and of class $\mathcal{C}^2$. Then if f is an affine function, $\{(X_n, \Delta X_n)\ ,\ 0 \le n \le N-1\}$ is a Markov process. Conversely, if this process is Markovian and the densities of the variables ξ_n are strictly positive a.e., then we must have $f'' = 0$.*

PROOF: Taking into account Property (1) in Lemma 3.2 the Markov property for the process $\{(X_n, \Delta X_n)\}$ is equivalent to the conditional independence

$$\{\xi_n\ ,\ 0 \le n \le p-1\} \underset{X_p, X_{p+1}}{\perp\!\!\!\perp} \{\xi_n\ ,\ p \le n \le N-2\}, \tag{3.6}$$

for all $2 \le p \le N-3$. Fix a value of p between 2 and $N-3$. Now we will apply Proposition 2.2 to $(Z_1^1, \ldots, Z_1^n) = (\xi_0, \ldots, \xi_{p-1})$, $(Z_2^1, \ldots, Z_2^m) = (\xi_p, \ldots, \xi_{N-2})$, $X = X_p$, and $Y = X_{p+1}$, and to the system (3.3). In view of Lemma 3.2 this system has a unique solution for all $(\xi_0, \ldots, \xi_{p-1}) \in I\!\!R^p$, and $(\xi_p, \ldots, \xi_{N-2}) \in I\!\!R^{N-p-1}$. We have to show that hypothesis **(H.1)** of Section 2 holds. To do this we first express the system (3.4), which determines g_1, as

$$A_p \begin{pmatrix} X_1 \\ \vdots \\ \vdots \\ X_p \end{pmatrix} + \begin{pmatrix} f(X_1) \\ \vdots \\ \vdots \\ f(X_p) \end{pmatrix} = \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{p-2} \\ \xi_{p-1} + X_{p+1} \end{pmatrix}$$

where

$$A_p = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

Differentiating with respect to ξ_i, $0 \le i \le p-1$, we obtain

$$A_p \begin{pmatrix} \frac{\partial X_1}{\partial \xi_i} \\ \vdots \\ \\ \vdots \\ \frac{\partial X_p}{\partial \xi_i} \end{pmatrix} + \begin{pmatrix} f'(X_1)\frac{\partial X_1}{\partial \xi_i} \\ \vdots \\ \\ \vdots \\ f'(X_p)\frac{\partial X_p}{\partial \xi_i} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

and we deduce that $\frac{\partial X_p}{\partial \xi_i}$ is the element in the last row and $(i+1)$–th column of the inverse matrix of

$$B_{1,p} = \begin{pmatrix} -2+f'(X_1) & 1 & & & \\ 1 & -2+f'(X_2) & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2+f'(X_{p-1}) & 1 \\ & & & 1 & -2+f'(X_p) \end{pmatrix}$$

That is,

$$\text{(3.7)} \qquad \frac{\partial X_p}{\partial \xi_0} = \frac{(-1)^{p+1}}{\det B_{1,p}}, \quad \frac{\partial X_p}{\partial \xi_i} = \frac{(-1)^{p+1+i} \det B_{1,i}}{\det B_{1,p}}, \quad i \geq 1.$$

As a consequence we obtain

$$\text{(3.8)} \qquad \frac{\partial X_p}{\partial X_{p+1}} = \frac{\partial X_p}{\partial \xi_{p-1}} = \frac{\det B_{1,p-1}}{\det B_{1,p}}.$$

We proceed similarly with the system (3.5) giving g_2:

$$A_{N-p-1} \begin{pmatrix} X_{p+1} \\ \vdots \\ \vdots \\ X_{N-1} \end{pmatrix} + \begin{pmatrix} f(X_{p+1}) \\ \vdots \\ \vdots \\ f(X_{N-1}) \end{pmatrix} = \begin{pmatrix} \xi_p - X_p \\ \xi_{p+1} \\ \vdots \\ \xi_{N-2} \end{pmatrix}$$

with

$$A_{N-p-1} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}.$$

Differentiating with respect to ξ_j, $p \leq j \leq N-2$, we obtain

$$A_{N-p-1} \begin{pmatrix} \frac{\partial X_{p+1}}{\partial \xi_j} \\ \vdots \\ \vdots \\ \frac{\partial X_{N-1}}{\partial \xi_j} \end{pmatrix} + \begin{pmatrix} f'(X_{p+1}) \frac{\partial X_{p+1}}{\partial \xi_j} \\ \vdots \\ \vdots \\ f'(X_{N-1}) \frac{\partial X_{N-1}}{\partial \xi_j} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

and $\frac{\partial X_{p+1}}{\partial \xi_j}$ is the element in the $(j-p+1)$–th column and first row of the inverse matrix of $B_{p+1,N-1}$, where this matrix is defined as $B_{1,p}$ but with the indexes of X running from $p+1$ to $N-1$ in the main diagonal. That is,

$$\frac{\partial X_{p+1}}{\partial \xi_{N-2}} = \frac{(-1)^{N-p}}{\det B_{p+1,N-1}},$$
$$\frac{\partial X_{p+1}}{\partial \xi_j} = \frac{(-1)^{j-p} \det B_{j+2,N-1}}{\det B_{p+1,N-1}}, \quad p \leq j \leq N-3. \tag{3.9}$$

As a consequence we obtain

$$\frac{\partial X_{p+1}}{\partial X_p} = -\frac{\partial X_{p+1}}{\partial \xi_p} = -\frac{\det B_{p+2,N-1}}{\det B_{p+1,N-1}}. \tag{3.10}$$

From (3.8) and (3.10) we get

$$1 - \frac{\partial X_p}{\partial X_{p+1}} \frac{\partial X_{p+1}}{\partial X_p} = 1 + \frac{\det B_{1,p-1} \det B_{p+2,N-1}}{\det B_{1,p} \det B_{p+1,N-1}} > 0, \tag{3.11}$$

and on the other hand, $\frac{\partial X_p}{\partial \xi_i}$ and $\frac{\partial X_{p+1}}{\partial \xi_j}$ are non zero for all $0 \leq i \leq p-1$, $p \leq j \leq N-2$, which implies **(H.1)**.

Suppose that f is an affine function. Then the expression appearing in (3.11) is a constant. Thus the factorization condition (2.5) holds, and, by Proposition 2.2 the conditional independency (3.6) is true.

Suppose, conversely, that (3.6) holds. Then, from Lemma 2.4 this implies, taking $i=0$, $k=1$, $j=N-2$ and $l=N-3$, that

$$\frac{\partial}{\partial X_{p+1}} \left[\frac{\partial X_p / \partial \xi_1}{\partial X_p / \partial \xi_0} \right] \cdot \frac{\partial}{\partial X_p} \left[\frac{\partial X_{p+1} / \partial \xi_{N-3}}{\partial X_{p+1} / \partial \xi_{N-2}} \right] = 0, \tag{3.12}$$

almost surely. From (3.7) and (3.9) we deduce

$$\frac{\partial X_p}{\partial \xi_1} \left(\frac{\partial X_p}{\partial \xi_0} \right)^{-1} = 2 - f'(X_1),$$

and

$$\frac{\partial X_{p+1}}{\partial \xi_{N-3}} \left(\frac{\partial X_{p+1}}{\partial \xi_{N-2}} \right)^{-1} = 2 - f'(X_{N-1}).$$

Substituting these expressions into (3.12) we get

$$f''(X_1) \frac{\partial X_1}{\partial X_{p+1}} f''(X_{N-1}) \frac{\partial X_{N-1}}{\partial X_{p+1}} = 0, \quad \text{a.s.}$$

Observe that the derivatives $\frac{\partial X_1}{\partial X_{p+1}}$ and $\frac{\partial X_{N-1}}{\partial X_p}$ never vanish. Indeed, proceding as before, one obtains

$$\frac{\partial X_1}{\partial X_{p+1}} = \frac{(-1)^p}{\det B_{1,p}}, \quad \text{and} \quad \frac{\partial X_{N-1}}{\partial X_{p+1}} = \frac{(-1)^p}{\det B_{p+1,N-1}}.$$

Consequently, we obtain that

$$f''(X_1)f''(X_{N-1}) = 0, \tag{3.13}$$

a.s. If f is not affine we can find an interval $]t_1, t_2[\subset \mathbb{R}$ such that $f''(t) \neq 0$, $\forall t \in]t_1, t_2[$. The mapping from $(\xi_0, \ldots, \xi_{N-2})$ to $(X_1, \ldots X_{N-1})$ is a $\mathcal{C}^1$–diffeomorphism of $\mathbb{R}^{N-1}$. Consequently, from our hypothesis on the law of the variables ξ_n we deduce that the support of the law of $(X_1, \ldots X_{N-1})$ is $\mathbb{R}^{N-1}$. So, with positive probability, we have that $X_1 \in]t_1, t_2[$ and $X_{N-1} \in]t_1, t_2[$, which is in contradiction with (3.13). □

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