The Annals of Probability
1995, Vol. 23, No. 3, 1262–1288

MARKOV FIELD PROPERTY OF STOCHASTIC DIFFERENTIAL EQUATIONS

BY AURELI ALABERT, MARCO FERRANTE AND DAVID NUALART[1]

Universitat Autònoma de Barcelona, Università degli Studi di Padova and Universitat de Barcelona

The purpose of this paper is to prove a characterization of the conditional independence of two independent random variables given a particular functional of them, in terms of a factorization property. As an application we discuss the Markov field property for solutions of stochastic differential equations with a boundary condition involving the values of the process at times $t = 0$ and $t = 1$.

1. Introduction. The study of stochastic differential equations with boundary conditions can be traced back at least to Kwarkernaak [9]. He considered an n-dimensional equation of the form

$$dX_t = AX_t\,dt + dW_t, \qquad 0 \le t \le T, \tag{1.1}$$

with periodic boundary condition $X_0 = X_T$, and studied problems of prediction, smoothing and filtering related to the solution process, which in this case can be written down explicitly.

One can consider the general stochastic boundary value problem

$$\begin{aligned} dX_t &= f(t, X_t)\,dt + \sum_{i=1}^{k} g_i(t, X_t) \circ dW_t^i, \\ &\qquad 0 \le t \le 1, \qquad h(X_0, X_1) = 0, \end{aligned} \tag{1.2}$$

where $W = \{W_t,\ 0 \le t \le 1\}$ is a k-dimensional Wiener process and f, g_i: $[0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ and h: $\mathbb{R}^{2n} \to \mathbb{R}^n$ are measurable and locally bounded functions.

In general, we cannot expect the solution to these types of equations to be adapted to the Wiener filtration, because of the boundary condition. For this reason the stochastic integral in (1.2) makes no sense in the framework of the classical Itô stochastic calculus. An anticipating stochastic integration theory was developed in the late 1980's by several authors (see, for instance, [12] and [17]), thus permitting the use of the formalism of stochastic differentials in the setting of new problems. In this sense, the circle in the diffusion term of (1.2) denotes the extended (anticipating) Stratonovich integral.

Received February 1994; revised October 1994.
[1]Supported by DGICYT Grant PB90-0452.
AMS 1991 *subject classifications*. 60H10, 60J15, 60H07.
Key words and phrases. Stochastic differential equations, Markov property, conditional independence, reciprocal Markov processes.

In [18], Ocone and Pardoux studied equation (1.2) with linear coefficients, that is,

$$\begin{aligned} f(t,x) &= Ax + a(t), \\ g_i(t,x) &= B_i x + b_i(t), \\ h(x,y) &= F_0 x + F_1 y - F, \end{aligned}$$

where $a(t), b_i(t)$ are d-dimensional processes and A, B_i, F_0, F_1, F are constant matrices of appropriate dimensions. Using the rules of the anticipating stochastic calculus, Ocone and Pardoux established the existence and uniqueness of the solution in a suitable class of processes and studied the Markov property of the solution in the following sense:

We say that a stochastic process $\{X_t, t \in [a,b]\}$ is a *Markov field* if for any $s, t \in [a,b]$, $s \le t$, the σ-fields $\sigma\{X_r, r \in [s,t]\}$ and $\sigma\{X_r, r \in [a,b] - (s,t)\}$ are conditionally independent given X_s and X_t.

In [18], it is proved that in some particular cases the solution of (1.2) with linear coefficients is a Markov field. For instance, the Markov field property holds if either $B_i = 0$ for $i = 1, \dots, k$ (Gaussian case) or $a = b_1 = \cdots = b_k = 0$ and the fundamental solution of the associated homogeneous equation is a diagonal matrix.

The paper by Ocone and Pardoux was followed by several works on other classes of equations, where existence and uniqueness results as well as conditions for a Markov-type property to hold were established.

Nualart and Pardoux [13] studied the case of a one-dimensional equation with nonlinear drift, constant diffusion coefficient and a general nonlinear boundary condition. Second order equations with Dirichlet or Neuman boundary conditions were considered by Nualart and Pardoux [11, 14, 15]. Donati-Martin [2] studied equations with nonlinear drift, linear diffusion term and linear boundary condition. In all those cases the main result states that the solution is a Markov (germ Markov, in the second order case) field if and only if the drift has a special form (affine if the diffusion term is constant). The method for proving these results is based on a change of measure argument that can be briefly outlined as follows.

First one considers the particular case where the nonlinear drift is replaced by zero or by a linear coefficient. The Markov property for the explicit solution of this equation is proved by direct arguments. Then, an extended version of Girsanov theorem (see [10]) is used to obtain a new probability under which the law of the particular solution coincides with the law of the solution to the original problem with a nonlinear drift. Finally, it is shown that the Markov property can be translated into a factorization property for the Radon–Nikodým derivative.

This procedure has also been applied to discuss the Markov properties of stochastic difference equations (see [4] and Ferrante and Nualart [8]), higher dimensional nonlinear equations with constant diffusion coefficient (Ferrante [6] and Ferrante and Nualart [7]) and stochastic partial differential equations ([3, 5, 16]).

The main difficulty involved in this method is that the Carleman–Fredholm determinant of a certain integral operator must be explicitly computed in order to obtain conditions for the factorization of the density.

In Alabert and Nualart [1] a new approach was proposed and applied to a second order nonlinear stochastic difference equation. This new approach led in a natural way to formulation of the following general question: Given two independent random variables Z_1 and Z_2, are they conditionally independent given some function $g(Z_1, Z_2)$? An analytical characterization of this conditional independence can be given in some situations (see [1], Proposition 2.2), using the co-area formula of geometric measure theory.

In this paper we present a generalization of the characterization theorem given in [1] which allows us to give shorter proofs to known results about Markov properties and to treat new problems for which the change of measure method does not seem to apply.

The organization of the paper is as follows. Section 2 is devoted to stating and proving the characterization theorem of conditional independence which will be used in the subsequent sections. More precisely, we address the following general problem: Given two independent sub-σ-fields $\mathscr{F}_1$, $\mathscr{F}_2$ of a probability space, and given two random variables X and Y determined as the solution of a system of the form

$$X = g_1(Y, \omega),$$
$$Y = g_2(X, \omega),$$

where $g_i(y, \cdot)$ is $\mathscr{F}_i$-measurable ($i = 1,2$), under what conditions on g_1 and g_2 are $\mathscr{F}_1$ and $\mathscr{F}_2$ conditionally independent given X and Y? This problem arises in a natural way when treating stochastic equations with boundary conditions. The result is given in Theorem 2.1 and the proof is based on a change of variable formula [see (2.3)].

In Section 3 we study the one-dimensional stochastic differential equation

$$(1.3) \quad dX_t = f(t, X_{t-})\mu(dt) + dW_t, \qquad 0 \le t \le 1, \qquad X_0 = \psi(X_1),$$

where μ is a finite positive measure on $[0,1]$ such that $\mu(\{0\}) = \mu(\{1\}) = 0$. Under some monotonicity assumptions on the functions f and ψ we show the existence of a unique solution.

In Section 4 we provide necessary and sufficient conditions for the problem (1.3) to lead to a Markov field, using the characterization theorem proved in Section 2. More precisely, in Theorem 4.1 it is proved that the process X given by (1.3) is a Markov field only in the following particular cases:

1. X_0 is given.
2. $f(t, x) = A(t)x + B(t)$ for all $x \in \mathbb{R}$ and $t \in \operatorname{supp}(\mu)$.
3. $X_0 = aX_1 + b$, $a \neq 0$, and f is of the form (2) except maybe at some point $t_0 \in \operatorname{supp}(\mu)$.

Then, two particular cases of (1.3) are discussed, and the results of Nualart and Pardoux [13] are recovered.

In Section 5 we treat the case of general drift and diffusion terms, reducing it to the case of a constant diffusion coefficient by means of a change of variables. In this way we provide a direct proof of the results on the Markov property obtained by Donati-Martin [2].

2. Characterization of conditional independence. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two independent sub-σ-fields of $\mathscr{F}$. Consider two functions g_1: $\mathbb{R} \times \Omega \to \mathbb{R}$ and g_2: $\mathbb{R} \times \Omega \to \mathbb{R}$ such that g_i is $\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}_i$-measurable, for $i = 1, 2$, and they satisfy the following conditions for some $\varepsilon_0 > 0$:

(H1) For every $x \in \mathbb{R}$ and $y \in \mathbb{R}$ the random variables $g_1(y, \cdot)$ and $g_2(x, \cdot)$ possess absolutely continuous distributions and the function

$$\delta(x, y) = \sup_{0 < \varepsilon < \varepsilon_0} \frac{1}{\varepsilon^2} P\{|x - g_1(y)| < \varepsilon, |y - g_2(x)| < \varepsilon\}$$

is locally integrable in $\mathbb{R}^2$.

(H2) For any $|\xi| < \varepsilon_0$, $|\eta| < \varepsilon_0$, the system

$$\begin{aligned} x - g_1(y, \omega) &= \xi, \\ y - g_2(x, \omega) &= \eta, \end{aligned} \tag{2.1}$$

has a unique solution $(x, y) \in \mathbb{R}^2$, for almost all $\omega \in \Omega$.

(H3) For almost all $\omega \in \Omega$, the functions $y \to g_1(y, \omega)$ and $x \to g_2(x, \omega)$ are continuously differentiable and there exists a nonnegative random variable H such that $E(H) < \infty$ and

$$\sup_{\substack{|y - g_2(x, \omega)| < \varepsilon_0 \\ |x - g_1(y, \omega)| < \varepsilon_0}} |1 - g_1'(y, \omega) g_2'(x, \omega)|^{-1} \le H(\omega) \quad \text{a.s.}$$

Hypothesis (H2) implies the existence of two random variables X and Y determined by the system

$$\begin{aligned} X(\omega) &= g_1(Y(\omega), \omega), \\ Y(\omega) &= g_2(X(\omega), \omega). \end{aligned} \tag{2.2}$$

THEOREM 2.1. *Let g_1 and g_2 be two functions satisfying the preceding hypotheses* (H1)–(H3). *Then the following statements are equivalent*:

(i) *$\mathscr{F}_1$ and $\mathscr{F}_2$ are conditionally independent given the random variables X, Y.*

(ii) *There exist two functions F_i: $\mathbb{R}^2 \times \Omega \to \mathbb{R}$, $i = 1, 2$, which are $\mathscr{B}(\mathbb{R}^2) \otimes \mathscr{F}_i$-measurable for $i = 1, 2$, such that*

$$|1 - g_1'(Y) g_2'(X)| = F_1(X, Y, \omega) F_2(X, Y, \omega) \quad \text{a.s.}$$

PROOF. Let G_1 and G_2 be two bounded nonnegative random variables such that G_i is $\mathscr{F}_i$-measurable, for $i = 1, 2$. Suppose that f: $\mathbb{R}^2 \to \mathbb{R}$ is a nonnegative continuous and bounded function. For any $x \in \mathbb{R}$ we will denote by $f_i(x, \cdot)$ the density of the law of $g_i(x)$, for $i = 1, 2$. For each $\varepsilon > 0$, define $\varphi^\varepsilon(z) = (1/(2\varepsilon))\mathbf{1}_{[-\varepsilon, \varepsilon]}(z)$. Set

$$J(x, y) = |1 - g_1'(y) g_2'(x)|^{-1}.$$

We will first show the equality

$$\begin{aligned} &E[G_1 G_2 J(X, Y) f(X, Y)] \\ &\quad = \int_{\mathbb{R}^2} E[G_1 | g_1(y) = x] f_1(y, x) E[G_2 | g_2(x) = y] \\ &\qquad \times f_2(x, y) f(x, y)\, dx\, dy. \end{aligned} \tag{2.3}$$

Actually, we will see that both members of this equality arise when we compute the limit of

$$\int_{\mathbb{R}^2} E[G_1 G_2 \varphi^\varepsilon(x - g_1(y)) \varphi^\varepsilon(y - g_2(x))] f(x, y)\, dx\, dy \tag{2.4}$$

as ε tends to zero in two different ways.

For any $\omega \in \Omega$ we introduce the mapping Φ_ω: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Phi_\omega(x, y) = (x - g_1(y, \omega), y - g_2(x, \omega)) = (\bar{x}, \bar{y}).$$

Notice that $\Phi_\omega(X(\omega), Y(\omega)) = (0, 0)$. Denote by $D_{\varepsilon_0}(\omega)$ the set

$$D_{\varepsilon_0}(\omega) := \{(x, y) \in \mathbb{R}^2 : |x - g_1(y, \omega)| < \varepsilon_0, |y - g_2(x, \omega)| < \varepsilon_0\}.$$

Hypotheses (H2) and (H3) imply that for almost all ω the mapping Φ_ω is a C^1-diffeomorphism from $D_{\varepsilon_0}(\omega)$ onto $(-\varepsilon_0, \varepsilon_0)^2$. Therefore, making the change of variable $(\bar{x}, \bar{y}) = \Phi_\omega(x, y)$, we obtain for any $\varepsilon < \varepsilon_0$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \varphi^\varepsilon(x - g_1(y)) \varphi^\varepsilon(y - g_2(x)) f(x, y)\, dx\, dy \\ &\quad = \int_{\mathbb{R}^2} \varphi^\varepsilon(\bar{x}) \varphi^\varepsilon(\bar{y}) J(\Phi_\omega^{-1}(\bar{x}, \bar{y})) f(\Phi_\omega^{-1}(\bar{x}, \bar{y}))\, d\bar{x}\, d\bar{y}. \end{aligned}$$

By continuity this converges to $J(X, Y) f(X, Y)$ as ε tends to zero. The convergence of the expectations follows by the dominated convergence theorem because from hypothesis (H3) we have

$$J(\Phi_\omega^{-1}(\bar{x}, \bar{y})) \le \sup_{\substack{|y - g_2(x, \omega)| < \varepsilon_0 \\ |x - g_1(y, \omega)| < \varepsilon_0}} |1 - g_1'(y) g_2'(x)|^{-1} \le H \in L^1(\Omega),$$

if $|\bar{x}| < \varepsilon_0$ and $|\bar{y}| < \varepsilon_0$.

Consequently, (2.4) converges to the left-hand side of (2.3). Let us now turn to the proof that the limit of (2.4) equals the right-hand side of (2.3). We can write

$$
\begin{aligned}
&E[G_1G_2\varphi^\varepsilon(x-g_1(y))\varphi^\varepsilon(y-g_2(x))]\\
&\quad= E[G_1\varphi^\varepsilon(x-g_1(y))]E[G_2\varphi^\varepsilon(y-g_2(x))]\\
&\quad= \left(\int_{\mathbb{R}}\varphi^\varepsilon(x-\alpha)E[G_1|g_1(y)=\alpha]f_1(y,\alpha)\,d\alpha\right)\\
&\qquad\times\left(\int_{\mathbb{R}}\varphi^\varepsilon(y-\beta)E[G_2|g_2(x)=\beta]f_2(x,\beta)\,d\beta\right).
\end{aligned}
$$

We are going to take the limit of both factors as ε tends to zero. For the first one, the Lebesgue differentiation theorem tell us that for any $y\in\mathbb{R}$ there exists a set N^y of zero Lebesgue measure such that for all $x\notin N^y$,

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}\varphi^\varepsilon(x-\alpha)E[G_1|g_1(y)=\alpha]f_1(y,\alpha)\,d\alpha\\
&\qquad= E[G_1|g_1(y)=x]f_1(y,x).
\end{aligned}
$$

Similarly, for the second integral, for each fixed $x\in\mathbb{R}$, there will be a set N^x of zero Lebesgue measure such that for all $y\notin N^x$,

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}\varphi^\varepsilon(y-\beta)E[G_2|g_2(x)=\beta]f_2(x,\beta)\,d\beta\\
&\qquad= E[G_2|g_2(x)=y]f_2(x,y).
\end{aligned}
$$

We conclude that, except on the set

$$N=\{(x,y)\colon x\in N^y \text{ or } y\in N^x\},$$

which has measure zero, we will have the convergence

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}E[G_1G_2\varphi^\varepsilon(x-g_1(y))\varphi^\varepsilon(y-g_2(x))]\\
&\qquad= E[G_1|g_1(y)=x]f_1(y,x)E[G_2|g_2(x)=y]f_2(x,y).
\end{aligned}
$$

The preceding equality provides the pointwise convergence of the integrand appearing in the expression (2.4). The corresponding convergence of the integral is derived through the dominated convergence theorem, using hypothesis (H1).

Consequently, (2.3) holds for any continuous and bounded function f and this equality easily extends to any measurable and bounded function f. Taking $f=\mathbf{1}_B$, where B is a set of zero Lebesgue measure, and putting $G_1=G_2=1$, we deduce from (2.3) that $P\{(X,Y)\in B\}=0$ because $J(X,Y)>0$ a.s. As a consequence, the law of (X,Y) is absolutely continuous with a density given by

$$f_{XY}(x,y)=\frac{f_1(y,x)f_2(x,y)}{E[J(x,y)|X=x,Y=y]}.$$

Therefore, (2.3) implies that

$$E[G_1G_2J(x,y)|X=x,Y=y]f_{XY}(x,y) = E[G_1|g_1(y)=x]f_1(y,x)E[G_2|g_2(x)=y]f_2(x,y) \tag{2.5}$$

almost surely with respect to the law of (X,Y). Putting $G_2 \equiv 1$ we obtain

$$E[G_1J(x,y)|X=x,Y=y]f_{XY}(x,y) = E[G_1|g_1(y)=x]f_1(y,x)f_2(x,y), \tag{2.6}$$

and with $G_1 \equiv 1$ we get

$$E[G_2J(x,y)|X=x,Y=y]f_{XY}(x,y) = E[G_2|g_2(x)=y]f_1(y,x)f_2(x,y). \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) yields

$$E[G_1G_2J(x,y)|X=x,Y=y]E[J(x,y)|X=x,Y=y] = E[G_1J(x,y)|X=x,Y=y]E[G_2J(x,y)|X=x,Y=y]. \tag{2.8}$$

Conditioning first on the bigger σ-fields $\sigma(X,Y)\vee\mathscr{F}_1$ and $\sigma(X,Y)\vee\mathscr{F}_2$ in the right-hand side of (2.8), we obtain

$$\begin{aligned} E[G_1G_2J(X,Y)|X,Y]E[J(X,Y)|X,Y] &= E[G_1E[J(X,Y)|X,Y,\mathscr{F}_1]|X,Y] \\ &\quad \times E[G_2E[J(X,Y)|X,Y,\mathscr{F}_2]|X,Y]. \end{aligned} \tag{2.9}$$

Suppose first that $\mathscr{F}_1$ and $\mathscr{F}_2$ are conditionally independent given X and Y. This allows us to write the equality (2.9) as follows:

$$\begin{aligned} &E[G_1G_2J(X,Y)E[J(X,Y)|X,Y]|X,Y] \\ &\quad = E[G_1G_2E[J(X,Y)|X,Y,\mathscr{F}_1]E[J(X,Y)|X,Y,\mathscr{F}_2]|X,Y]. \end{aligned}$$

Taking the expectation of both members of the above equality we obtain

$$J^{-1}(X,Y) = \frac{E[J(X,Y)|X,Y]}{E[J(X,Y)|X,Y,\mathscr{F}_1]E[J(X,Y)|X,Y,\mathscr{F}_2]}.$$

This implies the desired factorization because any random variable $\sigma(X,Y)\vee\mathscr{F}_i$-measurable $(i=1,2)$ can be written as $F(X(\omega),Y(\omega),\omega)$ for some $\mathscr{B}(\mathbb{R}^2)\times\mathscr{F}_i$-measurable function $F\colon \mathbb{R}^2\times\Omega\to\mathbb{R}$.

Conversely, suppose that (ii) holds. Then we have, from (2.9),

$$\begin{aligned} &E[G_1G_2|X,Y] \\ &\quad = E[G_1G_2F_1(X,Y)F_2(X,Y)J(X,Y)|X,Y] \\ &\quad = \frac{E[G_1F_1(X,Y)J(X,Y)|X,Y]E[G_2F_2(X,Y)J(X,Y)|X,Y]}{E[J(X,Y)|X,Y]}. \end{aligned}$$

Writing this equality for $G_1 \equiv 1$, $G_2 \equiv 1$ and for $G_1 \equiv G_2 \equiv 1$, we conclude

$$E[G_1G_2|X,Y] = E[G_1|X,Y]E[G_2|X,Y]. \qquad \square$$

REMARKS. Some of the conditions appearing in the above hypotheses can be weakened or modified and the conclusion of Theorem 2.1 will continue to hold. In particular, in hypothesis (H3) we can replace $H(\omega)$ by $H_1(\omega)H_2(\omega)$, with $H_i(\omega)$ $\mathscr{F}_i$-measurable for $i = 1, 2$, and assume only $H_1(\omega)H_2(\omega) < \infty$ a.s. In (H1) the local integrability of the function $\delta(x, y)$ holds if the densities $f_1(y, z)$ and $f_2(x, z)$ of $g_1(y)$ and $g_2(x)$ are locally bounded in $\mathbb{R}^2$.

The following two lemmas will be used in the application of the preceding factorization property. The proof of the first lemma is immediate.

LEMMA 2.1. *Suppose that* $a_1(\omega_1)a_2(\omega_2) = b_1(\omega_1)b_2(\omega_2)$ *a.e., on a product measure space* $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \otimes \mathscr{F}_2, \mu_1 \times \mu_2)$. *Then either*:

(i) $a_1 = 0$ μ_1*-a.e. and/or* $a_2 = 0$ μ_2*-a.e.*

or

(ii) *there is a constant* $k \neq 0$ *such that* $a_1(\omega_1) = kb_1(\omega_1)$ μ_1*-a.e. and* $a_2(\omega_2) = k^{-1}b_2(\omega_2)$ μ_2*-a.e.*

LEMMA 2.2. *Consider two independent σ-fields* $\mathscr{F}_1$, $\mathscr{F}_2$ *on a probability space* $(\Omega, \mathscr{F}, P)$, *and two random variables* G_1, G_2 *such that* G_i *is* $\mathscr{F}_i$*-measurable for* $i = 1, 2$. *The following statements are equivalent*:

(a) *There exist two random variables* H_1 *and* H_2 *such that* H_i *is* $\mathscr{F}_i$*-measurable*, $i = 1, 2$ *and*

$$1 - G_1G_2 = H_1H_2.$$

(b) G_1 *or* G_2 *is constant a.s.*

PROOF. The fact that (b) implies (a) is obvious. Let us show that (a) implies (b). We can assume that the underlying probability space is a product space $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \otimes \mathscr{F}_2, P_1 \times P_2)$. Property (a) implies that

$$\left[\tilde{G}_1(\tilde{\omega}_1) - G_1(\omega_1)\right]G_2(\omega_2) = \left[H_1(\omega_1) - \tilde{H}_1(\tilde{\omega}_1)\right]H_2(\omega_2),$$

where $\tilde{G}_1$ and $\tilde{H}_1$ are independent copies of G_1 and H_1 on some space $(\tilde{\Omega}_1, \tilde{\mathscr{F}}_1, \tilde{P}_1)$. Lemma 2.1 applied to the above equality implies either:

(A) $G_2 = 0$ a.s. and/or $\tilde{G}_1 - G_1 = 0$ a.s.

or

(B) $G_2 = kH_2$ for some constant $k \neq 0$.

Then (A) leads to property (b) directly and (B) implies that $1 = [H_1 + kG_1]H_2$. Applying again Lemma 2.1 to this identity yields that H_2 and thus G_2 is a.s. constant. □

COROLLARY 2.1. *Assume the hypotheses of Theorem* 2.1 *and, in addition, that* $1 - g_1'(Y)g_2'(X)$ *has constant sign. Then conditions* (i) *and* (ii) *of this theorem are equivalent to*:

(iii) *One* (*or both*) *of the variables* $g_1'(Y)$ *and* $g_2'(X)$ *is almost surely constant with respect to the conditional law given* X, Y.

3. Stochastic differential equations with boundary conditions. Let $(\Omega, \mathscr{F}, P)$ be the classical Wiener space. That is, $\Omega = C_0([0,1])$ is the space of continuous functions vanishing at zero, $\mathscr{F}$ is the Borel σ-field of Ω and P is the Wiener measure. The canonical process $W_t(\omega) = \omega(t)$ will be a standard Brownian motion.

We want to study the stochastic differential equation

$$
\begin{aligned}
X_t &= X_0 + \int_{(0,t]} f(s, X_{s-})\mu(ds) + \omega_t, \qquad 0 \le t \le 1, \\
X_0 &= \psi(X_1)
\end{aligned}
\tag{3.1}
$$

where μ is a finite measure on $[0,1]$, such that $\mu(\{0\}) = \mu(\{1\}) = 0$, and f and ψ are real-valued measurable functions on $[0,1] \times \mathbb{R}$ and $\mathbb{R}$, respectively. Note that this is a pathwise equation which is well formulated for each $\omega \in \Omega$.

Assuming suitable monotonicity assumptions on the functions f and ψ one can show that (3.1) has a unique solution. More precisely, we can establish the following theorem.

THEOREM 3.1. *Suppose that the functions* $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ *and* $\psi\colon \mathbb{R} \to \mathbb{R}$ *satisfy the following conditions.*

(a1) $|f(t,x) - f(t,y)| \le K|x-y|$, *for all* $t \in [0,1]$, $x, y \in \mathbb{R}$, *and for some constant* $K > 0$ *and* $\sup_{0 \le s \le 1} |f(s,0)| < \infty$.

(a2) *For each* t *such that* $\mu(\{t\}) > 0$, *the mapping* $x \to x + f(t,x)\mu(\{t\})$ *is nondecreasing.*

(b1) $\psi\colon \mathbb{R} \to \mathbb{R}$ *is a continuous and nonincreasing function.*

Then (3.1) *admits a unique solution for any* $\omega \in C_0([0,1])$ *which is a cadlag function.*

PROOF. For each $x \in \mathbb{R}$, let us denote by $\varphi_{s,t}(x)$ the unique solution to the equation

$$
\varphi_{s,t}(x) = x + \int_{(s,t]} f(r, \varphi_{s,r-}(x))\mu(dr) + \omega_t - \omega_s, \tag{3.2}
$$

with $0 \le s \le t \le 1$. The Lipschitz and linear growth conditions imposed on the function f insure that this equation has a unique solution. From hypotheses (a1) and (a2) it follows that the mapping $x \to \varphi_{s,t}(x)$ is continuous and nondecreasing. Indeed, the continuity property is clear, and by an approximation argument it suffices to show the monotonicity property when the measure μ has a finite number of jumps. In that case we deduce the nondecreasing property of the mapping $x \to \varphi_{s,t}(x)$ at a jump time t from the relation

$$
\varphi_{s,t}(x) = \varphi_{s,t-}(x) + f(t, \varphi_{s,t-}(x))\mu(\{t\}).
$$

In order to prove that (3.1) admits a unique solution, it suffices to show that the equation

$$
x_0 = \psi(\varphi_{0,1}(x_0))
$$

has a unique solution x_0. This is a consequence of the fact that the mapping $x \to \psi(\varphi_{0,1}(x))$ is nonincreasing, due to hypothesis (b1) and the preceding arguments. □

In the sequel we will impose the following stronger hypotheses on the function f:

(a1′) f and its partial derivative $\partial_2 f = \partial f / \partial x$ are continuous and $|\partial_2 f(t, x)| \le K$ for all $t \in [0, 1]$, $x \in \mathbb{R}$.

(a2′) $1 + \partial_2 f(t, x)\mu(\{t\}) \ge \gamma$ for all t, x and for some $\gamma > 0$.

In that case, we know (cf. Protter [20]) that the mapping $x \to \varphi_{s,t}(x)$ is differentiable and its derivative satisfies the linear equation

$$\varphi'_{s,t}(x) = 1 + \int_{(s,t]} \partial_2 f(r, \varphi_{s,r-}(x))\varphi'_{s,r-}(x)\mu(dr).$$

This equation admits a unique solution given by

$$\varphi'_{s,t}(x) = \exp\left[\int_{(s,t]} \partial_2 f(r, \varphi_{s,r-}(x))\mu^c(dr)\right] \times \prod_{s<\tau\le t} \left[1 + \partial_2 f(\tau, \varphi_{s,\tau-}(x))\mu(\{\tau\})\right],$$

where μ^c is the continuous part of μ. Hypotheses (a1′) and (a2′) imply that

$$\varphi'_{s,t}(x) \ge K_1 > 0 \tag{3.3}$$

for some constant $K_1 > 0$ given by, for some integer $m \ge 0$,

$$K_1 = \exp(-K\mu^c([0,1]) - 2K\mu^d([0,1]) + m \log \gamma). \tag{3.4}$$

Let $X = \{X_t, t \in [0, 1]\}$ be the unique solution of (3.1). We want to study the mapping $\omega \to X(\omega)$. We will denote by D^μ the class of functions x: $[0, 1] \to \mathbb{R}$ which are cadlag and such that x is continuous except on the countable set $N = \{\tau \in [0, 1]: \mu(\{\tau\}) > 0\}$. The space D^μ is a separable Banach space with the supremum norm, and the mapping $\omega \to X(\omega)$ is continuous from $C_0([0, 1])$ into D^μ. Indeed, it suffices to show that $X_0(\omega)$ is a continuous function of ω. We recall that $X_0(\omega)$ is determined by the equation

$$X_0(\omega) = \psi(\varphi_{0,1}(X_0(\omega))).$$

Then, using the continuity of the mappings $\varphi_{0,1}$ and ψ in the variables (x, ω) and x, respectively, we obtain that $X_0(\omega)$ depends continuously on ω.

The mapping X is a bijection between $C_0([0, 1])$ and the set

$$\mathscr{C} = \{x \in D^\mu: x_0 = \psi(x_1) \text{ and } \Delta x_t = f(t, x_{t-})\mu(\{t\}), \text{ for any } t \in N\}. \tag{3.5}$$

In fact, for any $x \in \mathscr{C}$ there is a unique $\omega \in C_0([0, 1])$ such that $X(\omega) = x$ given by

$$\omega_t = x_t - x_0 - \int_{[0,t]} f(s, x_{s-})\mu(ds).$$

It can be checked that the supremum and the infimum of two elements of $\mathscr{C}$ are still in $\mathscr{C}$.

PROPOSITION 3.1. *Suppose that the functions f and ψ satisfy hypotheses* (a1′), (a2′) *and* (b1). *Denote by $X = \{X_t,\ t \in [0,1]\}$ the solution of* (3.1). *Then, for every $0 < t \leq 1$ we have $\operatorname{Im} X_t = \mathbb{R}$ and $\operatorname{Im} X_0 = \operatorname{Im} \psi$. In particular this implies that the support of the law of X_t is the whole real line for each $t \in (0,1]$.*

PROOF. Fix $0 < t \leq 1$ and a real number ξ. We want to find ω such that $X_t(\omega) = \xi$. We will choose a path ω of the form

$$\omega_s = \theta s \mathbf{1}_{[0,t]}(s) + \theta t \mathbf{1}_{(t,1]}(s), \tag{3.6}$$

where $\theta \in \mathbb{R}$. Note first that, for an ω of this type, the trajectory $\{X_s(\omega),\ t \leq s \leq 1\}$, assuming $X_t(\omega) = \xi$, satisfies the equation

$$X_s(\omega) = \xi + \int_{(t,s]} f(r, X_{r-}(\omega))\mu(dr), \qquad t \leq s \leq 1.$$

Consequently, the value $X_0 = \psi(X_1(\omega))$ does not depend on θ. Consider the function $\{R_s(\theta),\ 0 \leq s \leq t\}$ solution of the equation

$$R_s(\theta) = X_0 + \int_{[0,s]} f(r, R_{\tau-}(\theta))\mu(dr) + s\theta, \qquad s \in [0,t].$$

Then it suffices to find θ such that $R_t(\theta) = \xi$. The continuous path ω of the form (3.6) with this value of θ will satisfy the required property.

The mapping $\theta \to R_t(\theta)$ is continuously differentiable and its derivative is given by

$$\frac{dR_t}{d\theta} = \int_0^t \exp\left[\int_{(s,t]} \partial_2 f(r, R_{r-}(\theta))\mu^c(dr)\right] \times \prod_{s<\tau\leq t} \left[1 + \partial_2 f(\tau, R_{\tau-}(\theta))\mu(\{\tau\})\right] ds.$$

By hypotheses (a1′) and (a2′) we have that

$$\frac{dR_t}{d\theta} \geq K_1 t > 0,$$

which implies that

$$\lim_{\theta \to \pm\infty} R_t(\theta) = \pm\infty.$$

Hence, the image of the mapping $\theta \to R_t(\theta)$ is the whole real line. The case $t = 0$ is treated in a similar form. □

4. Markov properties. Let $X = \{X_t,\ t \in [0,1]\}$ be the solution of (3.1). Our aim is to study Markov properties of the stochastic process X. More precisely we are interested in the following notion.

DEFINITION 4.1. A stochastic process $X = \{X_t,\ t \in [0,1]\}$ is said to be a *Markov field* or a *reciprocal Markov process* if for any $0 < s < t \le 1$ the σ-fields $\sigma\{X_r,\ r \in [s,t]\}$ and $\sigma\{X_r,\ r \notin (s,t)\}$ are conditionally independent given X_s and X_t.

We have not included the case $s = 0$ in our definition in order to avoid some technical difficulties in the application of the characterization theorem.

We will write

$$\mathscr{F}_1 \coprod_{\mathscr{F}_3} \mathscr{F}_2$$

to mean that the σ-fields $\mathscr{F}_1$ and $\mathscr{F}_2$ are conditionally independent given $\mathscr{F}_3$.

Fix $0 < s < t \le 1$. From the properties of conditional independence it holds that

$$\sigma\{X_r,\ r \in [s,t]\} \coprod_{X_s,\, X_t} \sigma\{X_r,\ r \notin (s,t)\} \tag{4.1}$$

if and only if

$$\mathscr{F}^i_{s,t} \coprod_{X_s,\, X_t} \mathscr{F}^e_{s,t}, \tag{4.2}$$

where

$$\mathscr{F}^i_{s,t} = \sigma\{W_r - W_t,\ s \le r \le t\},$$
$$\mathscr{F}^e_{s,t} = \sigma\{W_r,\ 0 \le r \le s;\ W_r - W_t,\ t \le r \le 1\}.$$

Indeed, (4.1) implies (4.2) because

$$\mathscr{F}^i_{s,t} \subset \sigma\{X_r,\ r \in [s,t]\}$$

and

$$\mathscr{F}^e_{s,t} \subset \sigma\{X_r,\ r \notin (s,t)\}.$$

Conversely, from the inclusions

$$\sigma\{X_r,\ r \in [s,t]\} \subset \mathscr{F}^i_{s,t} \vee \sigma\{X_s, X_t\}$$

and

$$\sigma\{X_r,\ r \notin (s,t)\} \subset \mathscr{F}^e_{s,t} \vee \sigma\{X_s, X_t\}$$

it follows that (4.2) implies (4.1).

With the notation of the previous section and assuming hypotheses (a1′), (a2′) and (b1), let us define the random functions $g_1, g_2 \colon \mathbb{R} \times \Omega \to \mathbb{R}$ by

$$\begin{aligned} g_1(y) &= \varphi_{s,t}(y), \\ g_2(x) &= \varphi_{0,s}(\psi(\varphi_{t,1}(x))). \end{aligned} \tag{4.3}$$

The proof of the conditional independence (4.2) will be based on the application of Corollary 2.1 to the functions g_1 and g_2. In order to check that these functions verify hypotheses (H1)–(H3), we will apply the techniques of the stochastic calculus of variations.

Let us denote by $\mathscr{S}$ the class of random variables of the form

$$F = f(W_{t_1}, \dots, W_{t_n}), \qquad 0 < t_1 < \cdots < t_n \le 1,\ f \in C_b^\infty(\mathbb{R}^n).$$

The elements of $\mathscr{S}$ are called smooth random variables. The derivative of a smooth random variable is the stochastic process defined by

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) \mathbf{1}_{[0, t_i]}(t).$$

For any $p \geq 1$ we denote by $\mathbb{D}^{1,p}$ the closure of $\mathscr{S}$ under the norm

$$\|F\|_{1,p}^p = E(|F|^p) + E\left(\left|\int_0^1 (D_t F)^2 \, dt\right|^{p/2}\right).$$

The derivative operator is continuous in the norm $\|\cdot\|_{1,2}$ from $\mathbb{D}^{1,2}$ into $L^2([0,1] \times \Omega)$. For any $h \in L^2([0,1])$ and $F \in \mathbb{D}^{1,2}$ we will write

$$D_h F = \int_0^1 h_t \, D_t F \, dt.$$

We will make use of the duality relation

$$E[D_h F] = E\left[F \int_0^1 h_t \, dW_t\right] \tag{4.4}$$

for all $F \in \mathbb{D}^{1,2}$, $h \in L^2([0,1])$.

PROPOSITION 4.1. *Suppose that the functions* f *and* ψ *satisfy the hypotheses* (a1′), (a2′) *and* (b1). *Assume also that* ψ *is continuously differentiable. Denote by* $X = \{X_t, t \in [0,1]\}$ *the solution of* (3.1). *Then the random functions* g_1 *and* g_2 *defined by* (4.3) *satisfy hypotheses* (H1)–(H3) *of Section* 2.

PROOF OF (H1). We claim that $g_1(y) \in \mathbb{D}^{1,2}$ and, for any $h \in L^2([0,1])$,

$$\begin{aligned} D_h \varphi_{s,t}(y) &= \int_s^t h_r \, dr \\ &\quad + \int_{(s,t]} \partial_2 f(r, \varphi_{s,r-}(y)) \, D_h \varphi_{s,r-}(y) \mu(dr). \end{aligned} \tag{4.5}$$

In fact, consider the Picard approximation to $\varphi_{s,t}(y)$ defined by

$$\varphi_{s,t}^{n+1}(y) = y + \int_{(s,t]} f(r, \varphi_{s,r-}^n(y)) \mu(dr) + \omega_t - \omega_s,$$

if $n \geq 0$, and $\varphi_{s,t}^0(y) = y$. The sequence $\varphi_{s,t}^n(y)$ converges in $L^2(\Omega)$ to $\varphi_{s,t}(y)$ for each $t \in [s,1]$. Recursively we can show that $\varphi_{s,t}^n(y)$ and $\varphi_{s,t-}^n(y)$ belong to $\mathbb{D}^{1,2}$ for all $t \in [s,1]$ and

$$D_h \varphi_{s,t}^{n+1}(y) = \int_s^t h_r \, dr + \int_{(s,t]} \partial_2 f(r, \varphi_{s,r-}^n(y)) \, D_h \varphi_{s,r-}^n(y) \mu(dr). \tag{4.6}$$

From (4.6) it follows that

$$\sup_n E\left(\|D\varphi^n_{s,t}(y)\|_2^2\right) < \infty \quad \text{and} \quad \sup_n E\left(\|D\varphi^n_{s,t-}(y)\|_2^2\right) < \infty$$

for all $0 \le s \le t \le 1$. This implies that $\varphi_{s,t}, \varphi_{s,t-} \in \mathbb{D}^{1,2}$ for all $0 \le s \le t \le 1$ and, taking the limit in (4.6), (4.5) holds.

From the linear equation (4.5) we deduce the following formula for the derivative of $g_1(y) = \varphi_{s,t}(y)$:

$$\begin{aligned} D_r g_1(y) = {} & \exp\left[\int_{(r,t]} \partial_2 f(u, \varphi_{s,u-}(y))\mu^c(du)\right] \\ & \times \prod_{r<\tau\le t} [1 + \partial_2 f(\tau, \varphi_{s,\tau-}(y))\mu(\{\tau\})] \mathbf{1}_{[s,t]}(r). \end{aligned}$$

Notice that $D_r\varphi_{s,t}(y) \ge K_1 > 0$ with $K_1 > 0$ given by (3.4) and for every $r \in [s,t]$. In a similar way we obtain

$$D_r g_2(x) = \begin{cases} \exp\left[\int_{(r,s]} \partial_2 f(u, \varphi_{0,u-}(\psi(\varphi_{t,1}(x))))\mu^c(du)\right] \\ \quad \times \prod_{r<\tau\le s} [1 + \partial_2 f(\tau, \varphi_{0,\tau-}(\psi(\varphi_{t,1}(x))))\mu(\{\tau\})], \\ \hfill \text{if } 0 \le r \le s, \\ \varphi'_{0,s}(\psi(\varphi_{t,1}(x)))\psi'(\varphi_{t,1}(x))\exp\left[\int_{(r,1]} \partial_2 f(u, \varphi_{t,u-}(x))\mu^c(du)\right] \\ \quad \times \prod_{r<\tau\le 1} [1 + \partial_2 f(\tau, \varphi_{0,\tau-}(\psi(\varphi_{t,1}(x))))\mu(\{\tau\})], \\ \hfill \text{if } t \le r \le 1, \\ 0, \qquad \text{if } r \in (s,t). \end{cases}$$

Again $D_r g_2(x) \ge K_1 > 0$ if $r \in [0,s]$. However, notice that $D_r g_2(x) \le 0$ if $t \le r \le 1$.

In order to compute the densities of the random variables $g_1(y)$ and $g_2(x)$ and to show the local integrability of $\delta(x,y)$ we proceed as follows. For every $\varepsilon > 0$ and $x \in \mathbb{R}$ we introduce the function $\psi_\varepsilon^x \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\psi_\varepsilon^x(z) = \begin{cases} 0, & \text{if } z \le x - \varepsilon, \\ (1/2\varepsilon)[z - (x - \varepsilon)], & \text{if } x - \varepsilon < z \le x + \varepsilon, \\ 1, & \text{if } z \ge x + \varepsilon. \end{cases}$$

Then taking the derivative of $\psi_\varepsilon^x(g_1(y))$ and projecting this derivative on the function $h = \mathbf{1}_{[s,t]}$ we obtain

$$D_h[\psi_\varepsilon^x(g_1(y))] = \frac{1}{2\varepsilon}\mathbf{1}_{\{|x - g_1(y)| < \varepsilon\}} D_h[g_1(y)].$$

We have $D_h[g_1(y)] \geq K_1(t-s)$. Consequently, taking expectations in the above equality and using the duality relation (4.4) we obtain

$$\begin{aligned}
&\frac{1}{2\varepsilon} P(|x - g_1(y)| < \varepsilon) \\
&\quad = E\left[\frac{D_h[\psi_\varepsilon^x(g_1(y))]}{D_h[g_1(y)]}\right] \\
&\quad \leq \frac{E[D_h[\psi_\varepsilon^x(g_1(y))]]}{K_1(t-s)} = \frac{E[(W_t - W_s)\psi_\varepsilon^x(g_1(y))]}{K_1(t-s)} \\
&\quad \leq \frac{1}{K_1\sqrt{t-s}}.
\end{aligned}$$

This implies the absolute continuity of the law of $g_1(y)$ and the uniform boundedness in the variables x, y of the above probability.

Similarly,

$$\frac{1}{2\varepsilon} P(|y - g_2(x)| < \varepsilon) \leq \frac{1}{K_1 s} E[D_{h'}(\psi_\varepsilon^y(g_2(x)))] \leq \frac{1}{K_1\sqrt{s}},$$

where $h' = \mathbf{1}_{[0,s]}$. So hypothesis (H1) holds. □

PROOF OF (H2). We can prove directly that for all $\omega \in \Omega$ the transformation

$$(x, y) \mapsto (x - g_1(y, \omega), y - g_2(x, \omega))$$

is bijective from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $(\bar{x}, \bar{y}) \in \mathbb{R}^2$. Set $x = \bar{x} + \varphi_{s,t}(y)$. It suffices to show that the mapping

$$y \mapsto_\Gamma \varphi_{0,s}(\psi(\varphi_{t,1}(\bar{x} + \varphi_{s,t}(y)))) + \bar{y}$$

has a unique fixed point, and this follows from

$$\begin{aligned}
\frac{d\Gamma}{dy} &= \varphi'_{0,s}(\psi(\varphi_{t,1}(\bar{x} + \varphi_{s,t}(y))))\psi'(\varphi_{t,1}(\bar{x} + \varphi_{s,t}(y))) \\
&\quad \times \varphi'_{t,1}(\bar{x} + \varphi_{s,t}(y))\varphi'_{s,t}(y) \leq 0.
\end{aligned}$$

□

PROOF OF (H3). We have

$$1 - g'_1(y)g'_2(x) = 1 - \varphi'_{s,t}(y)\varphi'_{0,s}(\psi(\varphi_{t,1}(x)))\psi'(\varphi_{t,1}(x))\varphi'_{t,1}(x) \geq 1. \quad \square$$

The proof of the proposition is now complete. □

THEOREM 4.1. *Let* f: $[0,1] \times \mathbb{R} \to \mathbb{R}$ *and* ψ: $\mathbb{R} \to \mathbb{R}$ *be two functions satisfying hypotheses* (a1′), (a2′) *and* (b1). *Assume also that* ψ *is continuously differentiable. Let* μ *be a finite measure on* $[0,1]$ *with* $\mu(\{0\}) = \mu(\{1\}) = 0$.

Then the unique solution of the boundary value problem (3.1) *is a reciprocal Markov process if and only if one of the following conditions is satisfied*:

(a) $\psi' \equiv 0$.

(b) *For each* $t \in [0,1] \cap \operatorname{supp}(\mu)$ *the function* $\partial_2 f(t,x)$ *does not depend on* x.

(c) ψ' *is constant and there exists* $t_0 \in (0,1) \cap \operatorname{supp}(\mu)$ *such that* $\partial_2 f(t,x)$ *does not depend on* x *for each* $t \in [0,1] \cap \operatorname{supp}(\mu)$, $t \neq t_0$.

PROOF. Fix $0 < s < t \leq 1$. The random variables X_s and X_t are solutions of the system

$$X_t = g_1(X_s),$$
$$X_s = g_2(X_t),$$

with $g_1(y) = \varphi_{s,t}(y)$ and $g_2(x) = \varphi_{0,s}(\psi(\varphi_{t,1}(x)))$. We have seen in Proposition 4.1 that Corollary 2.1 can be applied to these random functions. As a consequence,

$$\mathscr{F}^i_{s,t} \coprod_{X_s, X_t} \mathscr{F}^e_{s,t}$$

if and only if $g_1'(X_s)$ or $g_2'(X_t)$ is constant with respect to a regular version of the conditional probability given X_s and X_t.

For each path X in D^μ we define

$$G_X(I) = \exp\left[\int_I \partial_2 f(r, X_{r-})\mu^c(dr)\right] \prod_{\tau \in I} [1 + \partial_2 f(\tau, X_{\tau-})\mu(\{\tau\})].$$

We know that $G_X(I) \geq K_1 > 0$. We have

$$g_1'(X_s) = G_X((s,t]),$$

and

$$g_2'(X_t) = G_X([0,1] \setminus (s,t])\psi'(X_1).$$

Consequently, by Corollary 2.1 the conditional independence $\mathscr{F}^i_{s,t} \coprod_{X_s, X_t} \mathscr{F}^e_{s,t}$ holds if and only if there exists a measurable function $h\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$G_X((s,t]) = h(X_s, X_t) \quad \text{a.s.} \tag{4.7}$$

or

$$G_X([0,1] \setminus (s,t])\psi'(X_1) = h(X_s, X_t) \quad \text{a.s.} \tag{4.8}$$

If $\psi' \equiv 0$, then (4.8) clearly holds. Actually in this case X is a Markov process. So we will assume in the sequel that $\psi' \not\equiv 0$.

Let us start by proving that condition (b) or condition (c) implies that the solution of (3.1) is a reciprocal Markov process. It is easy to see that if (b) holds, then $G_X((s,t])$ is always constant and therefore (4.7) is true. Let us now assume that (c) holds. We have that

if $t_0 \notin (s,t]$, then $G_X((s,t])$ is constant, and

if $t_0 \in (s,t]$, then $G_X([0,1] \setminus (s,t])\psi'(X_1)$ is constant.

Consequently, for any $0 < s < t \le 1$, (4.7) or (4.8) is true.

To prove that the Markov property of the solution X of (3.1) implies one of the two conditions (b) or (c), we shall divide the exposition into four steps.

STEP 1. Let $x_1 < x_2$ be two real numbers. Fix $r \in (0,1)$. For each $\varepsilon > 0$ satisfying $[r-\varepsilon, r+\varepsilon] \subset (0,1)$, there exists a couple of paths $X^{1,\varepsilon}$ and $X^{2,\varepsilon}$ in the set of solutions $\mathscr{C}$ defined in (3.5) such that:

(i) $X^{1,\varepsilon} \equiv X^{2,\varepsilon}$ on $[0,1] \setminus (r-\varepsilon, r+\varepsilon)$.
(ii) $X^{1,\varepsilon}_{r-} = x_1$, $X^{2,\varepsilon}_{r-} = x_2$.
(iii) $X^{1,\varepsilon}_1 = X^{2,\varepsilon}_1 = x$, with $\psi'(x) \neq 0$.

PROOF OF STEP 1. Fix any $y \in \mathbb{R}$, and take $\xi^1 \in C_0([r+\varepsilon, 1])$ such that the solution at $t = 1$ of

$$X_t = y + \int_{(r+\varepsilon, t]} f(u, X_{u-})\mu(du) + \xi^1_t, \qquad t \in [r+\varepsilon, 1],$$

belongs to the open set $\{x \in \mathbb{R}: \psi'(x) \neq 0\}$. This is possible thanks to Proposition 3.1. Then, with $x = \psi(X_1)$ and another arbitrary point $z \in \mathbb{R}$, take $\xi^2 \in C_0([0, r-\varepsilon])$ such that

$$X_t = x + \int_{(0,t]} f(u, X_{u-})\mu(du) + \xi^2_t, \qquad t \in [0, r-\varepsilon],$$

give a solution with $X_{r-\varepsilon} = z$.

We can also find an element $\xi^3 \in C_0([r-\varepsilon, r])$ such that the solution of

$$X_t = z + \int_{(r-\varepsilon, t]} f(u, X_{u-})\mu(du) + \xi^3_t, \qquad t \in [r-\varepsilon, r]$$

satisfies $x_{r-} = x_1$, and an element $\xi^4 \in C_0([r, r+\varepsilon])$ which forces the solution of

$$X_t = x_1 + f(r, x_1)\mu(\{x_1\}) + \int_{(r, r+\varepsilon]} f(u, X_{u-})\mu(du) + \xi^4_t$$

to satisfy $X_{r+\varepsilon} = y$.

Now, clearly, the function $\omega \in C_0([0,1])$ defined by

$$\omega_u = \begin{cases} \xi^2_u, & u \in [0, r-\varepsilon], \\ \xi^2_{r-\varepsilon} + \xi^3_u, & u \in [r-\varepsilon, r], \\ \xi^2_{r-\varepsilon} + \xi^3_r + \xi^4_u, & u \in [r, r+\varepsilon], \\ \xi^2_{r-\varepsilon} + \xi^3_r + \xi^4_{r+\varepsilon} + \xi^1_u, & u \in [r+\varepsilon, 1], \end{cases}$$

produces a path $X^{1,\varepsilon} \equiv \{X^{1,\varepsilon}_u, u \in [0,1]\} \in \mathscr{C}$ and $X^{1,\varepsilon}_{r-} = x_1$.

In a similar way, and using the same functions ξ^1 and ξ^2, and the same real numbers x, y, z, we get a second path $X^{2,\varepsilon} \in \mathscr{C}$ with $X^{2,\varepsilon}_{r-} = x_2$. Clearly the functions $X^{1,\varepsilon}$ and $X^{2,\varepsilon}$ possess the desired properties. □

STEP 2. Assume $G_X((s,t]) = h(X_s, X_t)$ a.s. for some measurable function h. Then, for any $r \in (s,t) \cap \operatorname{supp}(\mu)$, $\partial_2 f(r,x)$ is constant in x.

PROOF OF STEP 2. Suppose that there exist $r \in (s,t) \cap \operatorname{supp}(\mu)$ and two real numbers x_1, x_2 such that

$$\partial_2 f(r, x_1) < \partial_2 f(r, x_2).$$

We are going to show that this leads to a contradiction. We will assume $x_1 < x_2$.

Let $\{X^{1,\varepsilon}, \varepsilon > 0\}$ and $\{X^{2,\varepsilon}, \varepsilon > 0\}$ be two families of solutions constructed by the method presented in Step 1. For each $\varepsilon > 0$ such that $(r-\varepsilon, r+\varepsilon) \subset (s,t)$, and for each $\eta > 0$, we can consider the open sets

$$\begin{aligned} B^1_{\varepsilon,\eta} &= \{X \in D^{\mu} : \|X - X^{1,\varepsilon}\|_\infty < \eta\}, \\ B^2_{\varepsilon,\eta} &= \{X \in D^{\mu} : \|X - X^{2,\varepsilon}\|_\infty < \eta\}. \end{aligned} \tag{4.9}$$

Clearly, $P(X^{-1}(B^1_{\varepsilon,\eta})) > 0$ and $P(X^{-1}(B^2_{\varepsilon,\eta})) > 0$ because $X^{-1}(B^1_{\varepsilon,\eta})$ and $X^{-1}(B^2_{\varepsilon,\eta})$ are nonempty open subsets of the Wiener space. Moreover

$$B^1_{\varepsilon,\eta} \cap B^2_{\varepsilon,\eta} = \varnothing$$

if η is small enough.

We claim that there exists $\varepsilon > 0$ and $\eta > 0$ such that

$$\{G_X((s,t]) : X \in B^1_{\varepsilon,\eta}\} \quad \text{and} \quad \{G_X((s,t]) : X \in B^2_{\varepsilon,\eta}\}$$

are disjoint sets of real numbers. In order to show this claim we distinguish two different cases:

CASE 1. Suppose that r is an atom of μ; that is, $\mu(\{r\}) > 0$. We can write

$$\begin{aligned} G_X((s,t]) &= G_X((s,t] \setminus (r-\varepsilon, r+\varepsilon]) \\ &\quad \times \exp\left[\int_{(r-\varepsilon, r+\varepsilon]} \partial_2 f(u, X_{u-})\,\mu^c(du)\right] \\ &\quad \times \prod_{\substack{\tau \in (r-\varepsilon, r+\varepsilon) \\ \tau \neq r}} [1 + \partial_2 f(\tau, X_{\tau-})\,\mu(\{\tau\})] \\ &\quad \times [1 + \partial_2 f(r, X_{r-})\,\mu(\{r\})]. \end{aligned} \tag{4.10}$$

We have

$$\limsup_{\eta \downarrow 0} \sup_{\varepsilon' < \varepsilon} \sup_{X \in B^i_{\varepsilon',\eta}} |\partial_2 f(r, X_{r-}) - \partial_2 f(r, x_i)| = 0,$$

for $i = 1, 2$. On the other hand, the second and third factors in the expression (4.10) converge to 1 as $\varepsilon \downarrow 0$, uniformly with respect to X. Finally, for each $\varepsilon > 0$, the first factor $G_X((s,t] \setminus (r-\varepsilon, r+\varepsilon])$ converges to the common value

$$G_{X^{1,\varepsilon}}((s,t] \setminus (r-\varepsilon, r+\varepsilon]) = G_{X^{2,\varepsilon}}((s,t] \setminus (r-\varepsilon, r+\varepsilon])$$

as $\eta \downarrow 0$. This allows us to conclude that the sets $\{G_X((s,t]) \colon X \in B^1_{\varepsilon,\eta}\}$ and $\{G_X((s,t]) \colon X \in B^2_{\varepsilon,\eta}\}$ are disjoint for some $\varepsilon > 0$ and some $\eta > 0$.

CASE 2. Suppose $\mu(\{r\}) = 0$. We have

$$G_X((s,t]) = G_X((s,t] \setminus (r-\varepsilon, r+\varepsilon])\exp\left[\int_{(r-\varepsilon,\ r+\varepsilon]} \partial_2 f(u, X_{u-})\mu^c(du)\right]$$
$$\times \prod_{\tau \in (r-\varepsilon, r+\varepsilon]} [1 + \partial_2 f(\tau, X_{\tau-})\mu(\{\tau\})].$$

Notice that $X_r = X_{r-}$ for any $X \in D^\mu$ because r is a continuity point of μ. Moreover, $X^{i,\varepsilon}_r = x_i$ for $i = 1,2$. Then for any $\alpha > 0$ we can find $0 < \delta < \varepsilon$ such that

$$\sup_{|u-r|<\delta} |X^{i,\varepsilon}_u - x_i| < \alpha \quad \text{for } i = 1,2.$$

We can assume that $r - \delta$ and $r + \delta$ are continuity points of μ. Now if we fix the paths $X^{i,\varepsilon}$ on the interval $[r-\delta, r+\delta]$, α will not change, and we can make $\varepsilon - \delta$ small enough in such a way that the quantities

$$\exp\left[\int_{(r-\varepsilon, r+\varepsilon]\setminus(r-\delta, r+\delta]} \partial_2 f(u, X_{u-})\mu^c(du)\right]$$
$$\times \prod_{\tau \in (r-\varepsilon, r+\varepsilon]\setminus(r-\delta, r+\delta]} [1 + \partial_2 f(\tau, X_{\tau-})\mu(\{\tau\})]$$

for $i = 1,2$, are arbitrary close to 1 uniformly with respect to $X \in B^i_{\varepsilon,\eta}$.

Taking into account that $\mu((r-\delta, r+\delta]) > 0$, we can take α, δ, $\varepsilon - \delta$ and η small enough (in that order) in such a way that the above claim holds. This completes the proof that for some ε and η the sets

$$\left\{G_X((s,t]) \colon X \in B^1_{\varepsilon,\eta}\right\} \quad \text{and} \quad \left\{G_X((s,t]) \colon X \in B^2_{\varepsilon,\eta}\right\}$$

are disjoint.

Notice that $X \to G_X((s,t])$ is a continuous functional on D^μ. We have shown that there exist two disjoint sets $I_1, I_2 \subset \mathbb{R}$ such that

$$G_X((s,t]) \in I_1 \quad \text{if } X \in B^1_{\varepsilon,\eta},$$
$$G_X((s,t]) \in I_2 \quad \text{if } X \in B^2_{\varepsilon,\eta}.$$

Set $X^{1,\varepsilon}_s = X^{2,\varepsilon}_s = z_1$ and $X^{1,\varepsilon}_t = X^{2,\varepsilon}_t = z_2$. Then we have

$$h(X_s, X_t) \in I_1 \quad \text{a.s. if } X \in B^1_{\varepsilon,\eta}$$

and

$$h(X_s, X_t) \in I_2 \quad \text{a.s. if } X \in B^2_{\varepsilon,\eta}.$$

Consider the mappings

$$C_0([0,1]) \to_X D^\mu \to_{\pi_{s,t}} \mathbb{R}^2,$$
$$\omega \mapsto X(\omega) \mapsto (X_s(\omega), X_t(\omega)).$$

We are going to see that $\pi_{s,t}(B^1_{\varepsilon,\eta})$ and $\pi_{s,t}(B^2_{\varepsilon,\eta})$ both contain a rectangle

$$R = (z_1 - \lambda, z_1 + \lambda) \times (z_2 - \lambda, z_2 + \lambda)$$

for some $\lambda \leq \eta$. This will imply that $R \subset h^{-1}(I_1) \cap h^{-1}(I_2)$ and consequently

$$X^{-1}\left(\pi_{s,t}^{-1}\left(h^{-1}(I_1) \cap h^{-1}(I_2)\right)\right) \neq \varnothing,$$

since R is open and X and $\pi_{s,t}$ are continuous. However, this is impossible, because $I_1 \cap I_2 = \varnothing$, and we will get a contradiction.

Therefore, we only need to show that for any $x \in (z_1 - \lambda, z_1 + \lambda)$ and any $y \in (z_2 - \lambda, z_2 + \lambda)$, there exists a path in $\mathscr{C} \cap B^1_{\varepsilon,\eta}$ and another in $\mathscr{C} \cap B^2_{\varepsilon,\eta}$, both satisfying $X_s = x$ and $X_t = y$. Let us show the existence of the one in $B^1_{\varepsilon,\eta}$. This path can be constructed starting from $X^{1,\varepsilon}$ and modifying this function to make its graph contain the points (s, x) and (t, y), while keeping it in the set $\mathscr{C} \cap B^1_{\varepsilon,\eta}$.

The new path will coincide with $X^{1,\varepsilon}$ except in small intervals centered at the points s and t. We are going to construct the piece of path in an interval of the form $[s, s + \delta]$. The other pieces of the path can be defined by a similar method.

Take a point $\tilde{s} > s$, and let $\tilde{z}_1 = X^{1,\varepsilon}_{\tilde{s}}$. Take $\lambda = \eta$. Fix $x \in (z_1 - \eta, z_1 + \eta)$ and let $[x_1, x_2]$ be an interval containing both x and z_1, and contained in $(z_1 - \eta, z_1 + \eta)$. We know there exist paths in $\mathscr{C}$ joining the points (s, x_1) and $(\tilde{s}, \tilde{z}_1)$, and the points (s, x_2) and $(\tilde{s}, \tilde{z}_1)$. Let us denote by $\underline{X}$ and $\overline{X}$ these two paths. We can assume that $\underline{X}_r \leq X^{1,\varepsilon}_r \leq \overline{X}_r$ for all $r \in [s, \tilde{s}]$; otherwise, we can replace $\underline{X}_r$ by $\inf\{\underline{X}_r, X^{1,\varepsilon}_r\}$ and $\overline{X}_r$ by $\sup\{\overline{X}_r, X^{1,\varepsilon}_r\}$. Taking into account that the trajectories are right-continuous, we can find δ small enough in order to have $s + \delta \leq \tilde{s}$ and

$$\underline{X}_r, \overline{X}_r \in \left[X^{1,\varepsilon}_r - \eta, X^{1,\varepsilon}_r + \eta\right] \quad \text{for all } r \in [s, s + \delta].$$

Set $s' = s + \delta$ and $z'_1 = X^{1,\varepsilon}_{s'}$. Clearly $\underline{X}_{s'} \leq z'_1 \leq \overline{X}_{s'}$. Now, take a path $Y \in \mathscr{C}$ verifying $Y_s = x$ and $Y_{s'} = z'_1$, and define

$$X_r = \inf\left\{\sup\{Y_r, \underline{X}_r\}, \overline{X}_r\right\} \quad \text{for all } r \in [s, s'].$$

This constitutes a path on $[s, s']$ verifying $X_s = x$ and $X_{s'} = z'_1$ and $X_r \in [X^{1,\varepsilon}_r - \eta, X^{1,\varepsilon}_r + \eta]$ for all $r \in [s, s']$.

For the construction of a path over some interval $[s', s]$ on the left of s, satisfying $X_{s'} = X^{1,\varepsilon}_{s'}$, $X_s = x$ and $X_r \in [X^{1,\varepsilon}_r - \eta, X^{1,\varepsilon}_r + \eta]$ for all $r \in [s', s]$, we proceed similarly, taking into account the fact that although the paths are not left-continuous, it holds that, for any $X.Y \in \mathscr{C}$,

$$|Y_{r-} - X_{r-}| \leq \frac{1}{\gamma}|Y_r - X_r| \quad \text{for all } r \in [0, 1].$$

This is easily proved from the relation

$$X_r = X_{r-} + f(r, X_{r-})\mu(\{r\}) \quad \text{for all } X \in \mathscr{C}.$$

In this case we have to take $\lambda = \gamma\eta$.

Finally, we perform the same construction around t and complete the whole path by letting $X_r = X_r^{1,\varepsilon}$ outside these small intervals. The path thus obtained satisfies $Y_s = x, Y_t = y$ and $Y \in B^1_{\varepsilon,\eta}$. Exactly the same reasoning provides another path Y with $Y_s = x$, $Y_t = y$ and $Y \in B^2_{\varepsilon,\eta}$, and Step 2 is proved. □

STEP 3. Assume $G_X([0,1] \setminus (s,t])\psi'(X_1) = h(X_s, X_t)$ a.s. for some measurable function h. Then, for any $r \in ((0,1) \setminus [s,t]) \cap \operatorname{supp}(\mu)$, $\partial_2 f(r,x)$ is constant in x.

PROOF OF STEP 3. Let $r \in (t,1)$, $r \in \operatorname{supp}(\mu)$. We want to prove that $\partial_2 f(r,x)$ is constant in x. Suppose this is false. Then there exist two real numbers x_1 and x_2 such that $\partial_2 f(r,x_1) < \partial_2 f(r,x_2)$.

Given $\varepsilon > 0$ such that $[r-\varepsilon, r+\varepsilon] \subset (t,1)$ we can construct two families of functions $\{X^{1,\varepsilon},\ \varepsilon > 0\}$ and $\{X^{2,\varepsilon},\ \varepsilon > 0\}$ by the technique of Step 1. Consider the open balls $B^1_{\varepsilon,\eta}$ and $B^2_{\varepsilon,\eta}$ defined by (4.9). We can show that

$$\left\{G_X([0,1] \setminus (s,t])\psi'(X_1);\ X \in B^1_{\varepsilon,\eta}\right\}$$

and

$$\left\{G_X([0,1] \setminus (s,t])\psi'(X_1);\ X \in B^2_{\varepsilon,\eta}\right\}$$

are disjoint sets of real numbers for $\varepsilon > 0$ and $\eta > 0$ small enough.

This can be done as in Step 2, with the only difference that the factor $\psi'(X_1)$ appears. Denote by x the common value of the paths $X^{1,\varepsilon}$ and $X^{2,\varepsilon}$ at time $t = 1$. Since $\psi'(x) \neq 0$ and ψ' is continuous, we can choose η such that for all $X \in B^1_{\varepsilon,\eta} \cup B^2_{\varepsilon,\eta}$ we have $\inf|\psi'(X_1)| > 0$. Then we can repeat the same arguments as in Step 2. The case $r \in (0,s)$ would be treated similarly. □

STEP 4. The function $f(r,x)$ is of the form $f(r,x) = a(r)x + b(r)$ for all $r \in [0,1] \cap \operatorname{supp}(\mu)$, except for at most one point $t_0 \in (0,1)$.

PROOF OF STEP 4. We will abbreviate by P[r] the property $f(r,x) = a(r)x + b(r)$. From (4.7), (4.8) and Steps 2 and 3 we get that, for any $0 < s < t < 1$, either P[r] holds for any $r \in (s,t) \cap \operatorname{supp}(\mu)$ or P[r] holds for any $r \in ((0,1) \setminus [s,t]) \cap \operatorname{supp}(\mu)$.

Suppose there exists $t_0 \in (0,1) \cap \operatorname{supp}(\mu)$ such that $P[t_0]$ does not hold. Then we conclude that P[r] holds for all $r \neq t_0$, $r \in (0,1) \cap \operatorname{supp}(\mu)$. In fact, given such r we can choose $s < r < t$ such that $t_0 \notin (s,t)$ and from the above dichotomy P[r] holds in $(s,t) \cap \operatorname{supp}(\mu)$. Notice that the set $\{r \in (0,1) \cap \operatorname{supp}(\mu)$: P[r] holds$\}$ is closed. Therefore, t_0 must be an isolated point of the support of μ. □

Finally it suffices to show that if there exists $t_0 \in (0,1) \cap \operatorname{supp}(\mu)$ for which $\partial_2 f(t_0, x)$ is not constant, then ψ' is constant. Choose s such that $0 < s < t_0 < 1$. Let us compute

$$G_X((s,1]) = \exp\left[\int_{(s,1]} a(r)\mu^c(dr)\right] \prod_{\substack{\tau\in(s,1]\\ \tau\neq t_0}} [1 + a(\tau)\mu(\{\tau\})] \times \left(1 + \partial_2 f(t_0, X_{t_0-})\mu(\{t_0\})\right)$$

and

$$G_X([0,s]) = \exp\left[\int_{(s,1]} a(r)\mu^c(dr)\right] \prod_{\tau\in[0,s]} [1 + a(\tau)\mu(\{\tau\})]\psi'(X_1).$$

The random variable $(1 + \partial_2 f(t_0, X_{t_0-})\mu(\{t_0\}))$ is not constant because $\mu(\{t_0\}) \neq 0$, the support of the law of X_{t_0-} is $\mathbb{R}$ and $\partial_2 f(t_0, x)$ is not constant. Consequently $\psi'(X_1)$ is constant a.s. and this implies that ψ' is constant. The proof is complete. □

Let us illustrate the preceeding result with two particular examples.

EXAMPLE 1. Suppose that μ is Lebesgue measure. Let us consider the following stochastic differential equation:

$$\begin{aligned} dX_t &= f(t, X_t)\,dt + dW_t, \qquad t \in [0,1], \\ X_0 &= \psi(X_1). \end{aligned} \tag{4.11}$$

This equation, with the boundary condition written in the form $X_0 = g(X_1 - X_0)$ has been studied in [13] making use of the technique of change of probability.

By Theorem 4.1, if f is of class $C^{0,1}([0,1] \times \mathbb{R})$, with a bounded partial derivative in x, and ψ is of class C^1 with $\psi' \le 0$, then the unique solution of (4.11) is a reciprocal Markov process if and only if either $\psi' \equiv 0$ or f is of the form

$$f(t,x) = a(t)x + b(t).$$

EXAMPLE 2. Suppose that μ is a point measure of the form $\mu = \delta_{t_0}$ with $t_0 \in (0,1)$. Clearly the function $f(t, x)$ is here defined just for $t = t_0$ and therefore we can simply write $f(x)$. Assume that f is differentiable, $f'(x) + 1 \ge \gamma$, $\gamma > 0$, for all $x \in \mathbb{R}$ and ψ is a decreasing C^1 function with $\psi' \le 0$. Then the unique solution of (3.1) is given by

$$X_t = X_0 + W_t + f(X_0 + W_{t_0})\mathbf{1}_{[t_0,1]}(t), \tag{4.12}$$

where X_0 is the unique solution of

$$X_0 = \psi\left(X_0 + W_1 + f(X_0 + W_{t_0})\right). \tag{4.13}$$

A simple application of Theorem 4.1 to this case gives the following result. The cadlag stochastic process X_t defined by (4.12) is a reciprocal Markov process if and only if one of the functions f or ψ is affine.

5. Equations with a general diffusion coefficient. In this section we will consider stochastic differential equations of the form

$$(5.1)\qquad \begin{aligned} X_t &= X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\circ dW_s, \qquad t \in [0,1],\\ X_0 &= \psi(X_1), \end{aligned}$$

where $\int_0^t \sigma(X_s)\circ dW_s$ denotes the generalized Stratonovich integral defined as follows (see [12]):

DEFINITION 5.1. A measurable process $u = \{u_t,\ t \in [0,1]\}$ such that $\int_0^1 |u_t|\,dt < \infty$ is said to be Stratonovich integrable if for any $t \in [0,1]$ the random variables

$$S_t^\pi = \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\left(\int_{t_i\wedge t}^{t_{i+1}\wedge t} u_s\,ds\right)(W_{t_{i+1}} - W_{t_i}),$$

where $\pi = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$, converge in probability to a limit S_t as $|\pi| = \max_i(t_{i+1} - t_i)$ tends to zero. The limit will be called the Stratonovich integral of u, and we will write $S_t = \int_0^t u_s \circ dW_s$.

In the case where ψ is an affine function, these equations have been studied by Donati-Martin [2], who gave a general result on existence and uniqueness of solutions and studied the Markov properties of the solution in the case of σ linear. We show here how these equations can be reduced to the case of a constant diffusion coefficient by means of a change of variable, thus permitting an easier analysis of the Markov properties.

By a solution to (5.1) we mean a continuous process $\{X_t,\ t \in [0,1]\}$ such that $\{\sigma(X_t),\ 0 \le t \le 1\}$ is Stratonovich integrable and (5.1) holds.

We first recall a version of the anticipating Itô formula for Stratonovich integrals. For any $p > 1$ we will set $\mathbb{L}^{1,p} = L^p([0,1], \mathbb{D}^{1,p})$. We will denote by $\mathbb{L}_C^{1,p}$ the set of processes $u \in \mathbb{L}^{1,p}$ such that there exists a version of the derivative Du satisfying the following properties:

1. The sets of $L^p(\Omega)$-valued functions
$$\{s \mapsto D_{s\vee t} u_{s\wedge t}\}_{t\in[0,1]}$$
and
$$\{s \mapsto D_{s\wedge t} u_{s\vee t}\}_{t\in[0,1]}$$
are equicontinuous.
2. $\operatorname{ess\,sup}_{(s,t)\in[0,1]^2} E[|D_s u_t|^p] < \infty$.

We will denote by $(\mathbb{L}_C^{1,p})_{\mathrm{loc}}$ the set of measurable processes u such that there exists a sequence $\{(u_n, \Omega_n)\}_{n\in\mathbb{N}}$ with $\Omega_n \uparrow \Omega$, $u_n \in \mathbb{L}_C^{1,p}$ and $u|_{\Omega_n} = u_{n|\Omega_n}$ a.s. for all n. Then we have the following change of variables formula, which is a localized version of [19], Theorem 5.3.

LEMMA 5.1. *Let I be an open interval (bounded or not) of the real line and let $G \in C^2(I)$. Let X, A, B be stochastic processes in $[0,1]$ satisfying $A \in L^2([0,1])$ a.s, $B, X \in (\mathbb{L}_C^{1,4})_{\text{loc}}$, X has continuous paths, $P(X \in I) = 1$ and*

$$X_t = X_0 + \int_0^t A_s\, ds + \int_0^t B_s \circ dW_s.$$

Then

$$G(X_t) = G(X_0) + \int_0^t G'(X_s) A_s\, ds + \int_0^t G'(X_s) B_s \circ dW_s.$$

The following lemma, whose proof is straightforward, establishes the stability of the set of continuous processes in $(\mathbb{L}_C^{1,4})_{\text{loc}}$ under composition with C^1 functions.

LEMMA 5.2. *Let X be a continuous process belonging to $(\mathbb{L}_C^{1,4})_{\text{loc}}$. Let σ be a continuously differentiable function. Then $\sigma(X_t)$ belongs to $(\mathbb{L}_C^{1,4})_{\text{loc}}$.*

We turn now to the change of variables in (5.1). The following result is an easy consequence of the previous Lemmas 5.1 and 5.2.

PROPOSITION 5.1. *Let I be an open interval and $\sigma\colon I \to (0, +\infty)$ a C^1 function. Fix $c \in I$ and define*

$$G(x) = \int_c^x \frac{1}{\sigma(y)}\, dy. \tag{5.2}$$

Then G is a C^2 diffeomorphism between I and an open interval $J \subset \mathbb{R}$. Let $b, \psi\colon I \to \mathbb{R}$ be C^1 functions and define $f, \varphi\colon J \to \mathbb{R}$ by

$$f(y) = \frac{b(G^{-1}(y))}{\sigma(G^{-1}(y))}, \qquad \varphi(y) = (G \circ \psi \circ G^{-1})(y).$$

Then a continuous stochastic process $X = \{X_t,\ t \in [0,1]\}$ with values on I, belonging to $(\mathbb{L}_C^{1,4})_{\text{loc}}$, satisfies

$$\begin{aligned} dX_t &= b(X_t)\, dt + \sigma(X_t) \circ dW_t, \qquad t \in [0,1], \\ X_0 &= \psi(X_1), \end{aligned}$$

if and only if the process $Y_t = G(X_t)$, which also belongs to $(\mathbb{L}_C^{1,4})_{\text{loc}}$, satisfies

$$\begin{aligned} dY_t &= f(Y_t)\, dt + dW_t, \qquad t \in [0,1], \\ Y_0 &= \varphi(Y_1). \end{aligned} \tag{5.3}$$

We will finally discuss the Markov property in two particular cases.

CASE 1. Suppose that $\sigma > 0$ on $\mathbb{R}$ and

$$\int_0^{\infty} \frac{1}{\sigma(y)}\,dy = +\infty \quad \text{and} \quad \int_{-\infty}^{0} \frac{1}{\sigma(y)}\,dy = +\infty.$$

Then the function G defined in (5.2) maps $\mathbb{R}$ onto $\mathbb{R}$ and using Proposition 5.1 we get the following result.

THEOREM 5.1. *Let $b: \mathbb{R} \to \mathbb{R}$ and $\sigma: \mathbb{R} \to \mathbb{R}$ be two C^1 functions satisfying*:

(i) $\sigma > 0$ *and the function* $G(x) = \int_c^x (dy/\sigma(y))$ *maps* $\mathbb{R}$ *onto* $\mathbb{R}$.
(ii) *The function*

$$f(y) = \frac{b(G^{-1}(y))}{\sigma(G^{-1}(y))}$$

has a bounded derivative.

Let $\psi: \mathbb{R} \to \mathbb{R}$ be a C^1 nonincreasing function. Then, the boundary value problem (5.1) *has a unique solution in* $(\mathbb{L}_C^{1,4})_{\mathrm{loc}}$.

PROOF. Consider the system

$$\begin{aligned} dY_t &= f(Y_t)\,dt + dW_t, \qquad t \in [0,1], \\ Y_0 &= \varphi(Y_1), \end{aligned}$$

with $f(y) = (b \circ G^{-1})/(\sigma \circ G^{-1})$ and $\varphi = G \circ \psi \circ G^{-1}$. The function φ is continuously differentiable and nonincreasing. We know that in this situation there exists a unique solution $\{Y_t, t \in [0,1]\}$, and one can show that it belongs to $(\mathbb{L}_C^{1,4})_{\mathrm{loc}}$. Therefore, $X_t = G^{-1}(Y_t)$ is the unique solution to (5.1). □

Concerning the Markov property, we obtain:

THEOREM 5.2. *Under the hypotheses of Theorem* 5.1, *the solution to* (5.1) *is a Markov field if and only if*

(i) $\psi' \equiv 0$

or

(ii) $b(x) = A\sigma(x) + B\sigma(x)\int_c^x (1/\sigma(t))\,dt$, *for some* $A, B, c \in \mathbb{R}$.

PROOF. First notice that $\{X_t, t \in [0,1]\}$ is a Markov field if and only if the solution $\{Y_t, t \in [0,1]\}$ of (5.3) is so. Consider the system (5.3). Applying Theorem 4.1 to this case we obtain that $\{Y_t, t \in [0,1]\}$ is a Markov field if and only if $\varphi' \equiv 0$ or f' is constant.

The condition $\varphi' \equiv 0$ is clearly equivalent to $\psi' \equiv 0$. On the other hand, a simple computation shows that f' is constant if and only if $b' - (\sigma'/\sigma)b$ is constant. Solving this linear differential equation for b, we get

$$b(x) = \sigma(x)\left[\frac{b(c)}{\sigma(c)} + k\int_c^x \frac{1}{\sigma(t)}\,dt\right], \qquad c \in \mathbb{R}.$$

That is,

$$b(x) = A\sigma(x) + B\sigma(x)\int_c^x \frac{1}{\sigma(t)}\,dt \quad \text{for some } A, B, c \in \mathbb{R}. \qquad \square$$

CASE 2. We turn to the case σ linear $[\sigma(x) = \sigma x]$ and ψ affine. In particular, let $\psi(x) = (h - F_1 x)/F_0$, where $h, F_0, F_1 > 0$ and let $\sigma > 0$.

We can consider the equation

$$(5.4) \qquad \begin{aligned} &Y_t = Y_0 + \int_0^t f(Y_s)\,ds + W_t, \qquad t \in [0,1], \\ &F_0 \exp(\sigma Y_0) + F_1 \exp(\sigma Y_1) = h. \end{aligned}$$

If f is a C^1 function with a bounded derivative, there is a unique solution of this equation. In fact, the boundary condition can be written as $Y_1 = g(Y_0)$ with

$$g(y) = \frac{1}{\sigma}\log\left(\frac{h - F_0 e^{\sigma y}}{F_1}\right).$$

This g is a decreasing bijection between $(-\infty, (1/\sigma)\log(h/F_0))$ and $(-\infty, (1/\sigma)\log(h/F_1))$. On the other hand, the solution of $Y_t = Y_0 + \int_0^t f(Y_s)\,ds + W_t$ at time $t = 1$ as a function of Y_0 has a derivative bounded from below by a positive constant. This implies the existence of a unique solution.

Equations (5.4) and (5.1) are related through the change of variable $G(x) = \sigma^{-1}\log x$ and $b = \sigma x f(G(x))$. Applying Proposition 5.1, we deduce that $X_t = G^{-1}(Y_t) = \exp(\sigma Y_t)$ is a positive solution to (5.1).

In Donati-Martin [2] it is proved that, assuming

$$-1 < b'(x) - \frac{b(x)}{x} < 1, \qquad x > 0,$$

(5.1) has a unique solution, which takes values on $\mathbb{R}^+ \backslash \{0\}$. This unique solution must coincide with $G^{-1}(Y_t)$. Notice that the above condition on b is equivalent to $|f'| < 1$.

We know that $\{Y_t, t \in [0,1]\}$ is a Markov field if and only if f' is constant. Thus, reasoning as in the proof of Theorem 5.2, we easily obtain the result proved in [2], that is, that $\{X_t, t \in [0,1]\}$ is a Markov field if and only if

$$b(x) = Ax + Bx\log x, \qquad x > 0.$$

REFERENCES

[1] ALABERT, A. and NUALART, D. (1992). Some remarks on the conditional independence and the Markov property. In *Stochastic Analysis and Related Topics. Progress in Probability* **31** 343–363. Birkhäuser, Boston.

[2] DONATI-MARTIN, C. (1991). Equations différentielles stochastiques dans $\mathbb{R}$ avec conditions aux bords. *Stochastics Stochastics Rep.* **35** 143–173.

[3] DONATI-MARTIN, C. (1992). Quasi-linear elliptic stochastic partial differential equations. Markov property. *Stochastics Stochastics Rep.* **41** 219–240.

[4] DONATI-MARTIN, C. (1993). Propriété de Markov des équations stationnaires discrètes quasi-linéaires. *Stochastic Process. Appl.* **48** 61–84.

[5] DONATI-MARTIN, C. and NUALART, D. (1994). Markov property for elliptic stochastic partial differential equations. *Stochastics Stochastics Rep.* **46** 107–115.

[6] FERRANTE, M. (1993). Triangular stochastic differential equations with boundary conditions. *Rend. Sem. Mat. Univ. Padova* **90** 159–188.

[7] FERRANTE, M. and NUALART, D. (1994). Markov field property for stochastic differential equations with boundary conditions. *Stochastics Stochastic Rep.* To appear.

[8] FERRANTE, M. and NUALART, D. (1994). On the Markov property of stochastic difference equations. *Stochastic Process. Appl.* **52** 239–250.

[9] KWARKERNAAK, H. (1975). Periodic linear differential stochastic processes. *SIAM J. Control* **13** 400–413.

[10] KUSUOKA, S. (1982). The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity I. *J. Fac. Science Univ. Tokyo Sect. IA Math.* **29** 567–597.

[11] NUALART, D. (1991). Nonlinear transformations of the Wiener measure and applications. In *Stochastic Analysis: Liber Amicorum for Moshe Zakai* 397–432. Academic Press, San Diego.

[12] NUALART, D. and PARDOUX, E. (1988). Stochastic calculus with anticipating integrands. *Prob. Theory Related Fields* **78** 535–581.

[13] NUALART, D. and PARDOUX E. (1991). Boundary value problems for stochastic differential equations. *Ann. Probab.* **19** 1118–1144.

[14] NUALART, D. and PARDOUX, E. (1991). Second order stochastic differential equations with Dirichlet boundary conditions. *Stochastic Process. Appl.* **39** 1–24.

[15] NUALART, D. and PARDOUX, E. (1991). Stochastic differential equations with boundary conditions. In *Stochastic Analysis and Applications* (A. B. Cruzeiro and J. C. Zambrini, eds.). *Progress in Probability* **26** 155–175. Birkhäuser, Boston.

[16] NUALART, D. and PARDOUX, E. (1994). Markov field properties of solutions of white noise driven quasi-linear parabolic pde's. *Stochastics Stochastics Rep.* **48** 17–44.

[17] NUALART, D. and ZAKAI, M. (1986). Generalized stochastic integrals and the Malliavin calculus. *Probab. Theory Related Fields* **73** 255–280.

[18] OCONE, D. and PARDOUX, E. (1989). Linear stochastic differential equations with boundary conditions. *Probab. Theory Related Fields* **82** 439–526.

[19] PARDOUX, E. (1990). Applications of anticipating stochastic calculus to stochastic differential equations. *Stochastic Analysis and Related Topics II. Lecture Notes in Math.* **1444** 63–105. Springer, Berlin.

[20] PROTTER, P. (1990). *Stochastic Integration and Differential Equations. Applied Math.* **21.** Springer, Berlin.

AURELI ALABERT
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
08193 BELLATERRA
SPAIN

MARCO FERRANTE
DIPARTAMENTO DI MATEMATICA
PURA ED APPLICATA
VIA BELZONI 7
35100 PADOVA
ITALY

DAVID NUALART
FACULTAT DE MATEMÀTIQUES
UNIVERSITAT DE BARCELONA
GRAN VIA 585
08007 BARCELONA
SPAIN