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# A second-order Stratonovich differential equation with boundary conditions

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**Abstract**

In this paper we show that the solution of a second-order stochastic differential equation with diffusion coefficient $\sigma \dot{X}_t$ and boundary conditions $X_0 = 0$ and $X_1 = 1$ is a 2-Markov field if and only if the drift is a linear function. The proof is based on the method of change of probability and makes use of the techniques of Malliavin calculus.

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## 1. Introduction

In this paper we study the stochastic differential equation with boundary conditions

$$\begin{aligned} &\ddot{X}_t = f(X_t) + \sigma \dot{X}_t \circ \dot{W}_t, \quad 0 \leqslant t \leqslant 1, \\ &X_0 = 0,\ X_1 = 1, \end{aligned} \tag{1.1}$$

where $\{W_t, t \in [0,1]\}$ is a standard Wiener process, $\sigma > 0$ is a constant, and $f:\mathbb{R} \to \mathbb{R}$ is a continuous function. Because of the condition on $X_1$, we cannot expect to have a solution to (1.1) adapted to the Wiener process $W$. We will use the following notion of extended Stratonovich integral (cf. Nualart and Pardoux, 1988).

**Definition 1.1.** Let $u = \{u_s, s \in [0,T]\}$ be a measurable process such that $\int_0^T |u_s| \, ds < \infty$ with probability 1. Then we will say that $u$ is Stratonovich integrable

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on $[0, T]$ if the following limit exists in probability:

$$\int_0^T u_s \circ \mathrm{d}W_s := \lim_{|\pi| \searrow 0} \sum_{j=0}^{n-1} \left( \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} u_s \, \mathrm{d}s \right) (W_{t_{j+1}} - W_{t_j}),$$

where $\pi = \{0 = t_0 < \cdots < t_n = T\}$ runs over all partitions of $[0, T]$.

By a solution to (1.1) we mean a real-valued process $\{X_s, s \in [0, 1]\}$ of class $C^1$ such that $\{\sigma \dot{X}_s, s \in [0, 1]\}$ is Stratonovich integrable on $[0, t]$ for each $t \in [0, 1]$, and satisfies the system

$$\begin{aligned} \dot{X}_t &= \dot{X}_0 + \int_0^t f(X_s) \, \mathrm{d}s + \int_0^t \sigma \dot{X}_s \circ \mathrm{d}W_s, \quad 0 \leqslant t \leqslant 1, \\ X_0 &= 0, \; X_1 = 1. \end{aligned} \tag{1.2}$$

The values 0 and 1 at the boundary do not play a special role, and they can be replaced by any two constants $a \neq b$. In fact, if $Z_t$ is a solution to our equation with $Z_0 = a$, $Z_1 = b$, then $X_t = (Z_t - a)/(b - a)$ solves (1.1) with $f(x)$ replaced by $\bar{f}(x) = [1/(b-a)] \, f((b-a)x + a)$.

Several types of stochastic differential equations with boundary conditions have already been studied (see, for instance, Alabert, 1995 and the references therein). In particular, equations involving second order derivatives have been considered in Nualart and Pardoux (1991), Nualart (1991), Donati-Martin (1992), Donati-Martin and Nualart (1993) and Nualart and Pardoux (1994). In all these cases, the perturbation is in the form of an additive white noise, and the main result is that the solution has a Markov property if and only if the drift is an affine function. The proofs make use of an anticipating version of Girsanov Theorem.

Our motivation to study boundary value problems of the form (1.1) is twofold. On the one hand, we would like to test how the technique of the change of probability behaves when the diffusion coefficient is non-constant. For first-order equations with a linear diffusion coefficient, this has been done by Donati-Martin (1991). In our case, we also consider a linear coefficient but depending only on $\dot{X}_t$.

On the other hand, an equation of the type (1.1) can be regarded as a one-dimensional version of an elliptic s.p.d.e. of the form

$$\begin{aligned} \Delta X_z &= f(X_z) + \nabla X_z \cdot \dot{W}_z, \quad z \in D \subset \mathbb{R}^k, \\ X_{z|_{\partial D}} &= \varphi(z), \end{aligned} \tag{1.3}$$

$k = 1, 2, 3$. For $k > 1$, this is a difficult problem; it is not even clear how to formulate a notion of solution. The case of (1.3) with additive noise is studied in Donati-Martin (1992) and Donati-Martin and Nualart (1994).

Concerning the Markov properties of the solution, recall that if we state an initial condition $(\dot{X}_0, X_0)$, the adapted solution to (1.1) is 2-Markov in the sense of Russek (1980); that is, $\{(X_t, \dot{X}_t), t \in [0, 1]\}$ is a Markov process. Then, a natural question to ask is what type of Markov property is satisfied by this process when we impose $X_0 = 0$, $X_1 = 1$. We will consider the Markov field property, which can be stated as follows.

**Definition 1.2.** A stochastic process $\{Z_u, u \in [0, 1]\}$ is a *Markov field* if and only if for all $0 \leqslant s < t \leqslant 1$, the families $\{Z_u, u \in [s, t]\}$ and $\{Z_u, u \in ]s, t[^c\}$ are conditionally independent given $Z_s$ and $Z_t$.

We recall that this notion is weaker than the usual definition of Markov process. Markov fields are also called reciprocal processes (see Krener et al., 1990). In the particular cases of stochastic boundary value problems so far studied, the Markov field property only holds under rather restrictive conditions on the coefficients. We will prove in this paper that if Eq. (1.1) gives rise to a Markov field, then $f$ must be an affine function, and that in such a case the solution is in fact a Markov process. The method employed to prove this result is based in a change of measure in Wiener space induced by a transformation of the form

$$T(\omega) = \omega + \int_0^{\cdot} u_s(\omega)\,ds. \tag{1.4}$$

The same transformation is used to obtain an existence and uniqueness result for (1.1), starting from the solution when $f \equiv 0$. We refer to Alabert (1995) for a survey of the main features of this change of probability method. Another alternative procedure has been suggested in Alabert et al. (1995), but it does not seem to be suitable for the equation considered here.

Section 2 is devoted to some preliminaries on the analysis in Wiener space. In Section 3 we tackle the problem of existence and uniqueness of a solution to (1.1). A unique solution exists under monotonicity conditions on $f$. Section 4 is concerned with the Markov field property for the process $\{(X_t, \dot{X}_t), t \in [0, 1]\}$, where $X_t$ is the solution to (1.1) found in Section 3.

## 2. Preliminaries

Let $(\Omega, H_0, P)$ be the classical Wiener space: $\Omega = C_0([0, 1])$ is the Banach space of continuous functions on $[0, 1]$ vanishing at zero, equipped with the supremum norm and the associated Borel $\sigma$-field; $H_0$ is the Hilbert space of functions in $\Omega$ with derivatives in $L^2([0, 1])$, with the inner product $\langle h, g\rangle_{H_0} := \langle \dot{h}, \dot{g}\rangle_{L^2}$; and $P$ is the standard Wiener measure. We will denote $H = L^2([0, 1])$.

Let $E$ be a real separable Hilbert space. A *smooth E-valued functional on* $\Omega$ is a random variable $F: \Omega \to E$ of the form

$$F = \sum_{j=1}^{m} f_j\left(\int_0^1 h_1(t)\,dW_t, \ldots, \int_0^1 h_n(t)\,dW_t\right)e_j,$$

where $h_i \in H$, $e_j \in E$, and $f_j$ are $C^\infty$ functions on $\mathbb{R}^n$ which have polynomial growth, together with all their derivatives. Denote by $\mathscr{S}(E)$ the set of these functionals.

For $F \in \mathscr{S}(E)$, we define its derivative $DF$ as the stochastic process $\{D_t F, 0 \leqslant t \leqslant 1\}$ given by

$$D_t F = \sum_{j=1}^{m} \sum_{i=1}^{n} \partial_i f_j\left(\int_0^1 h_1(s)\,dW_s, \ldots, \int_0^1 h_n(s)\,dW_s\right)h_i(t)e_j.$$

It can be shown that $F \in \mathscr{S}(E)$ implies $F \in L^p(\Omega; E)$ and $DF \in L^p(\Omega; H \otimes E)$, for all $p \geqslant 1$, and that $\mathscr{S}(E)$ is dense in $L^p(\Omega; E)$ (see Ikeda and Watanabe, 1989, Remark 8.2). Moreover, the operator

$$D: L^p(\Omega; E) \to L^p(\Omega; H \otimes E),$$

with domain $\mathcal{S}(E)$, is closable. Denoting by $\mathbb{D}^{1,p}(E)$ the closure of $\mathcal{S}(E)$ under the graph norm

$$\| F \|_{\mathbb{D}^{1,p}(E)} := \| F \|_{L^p(\Omega; E)} + \| DF \|_{L^p(\Omega; H \otimes E)},$$

we obtain a continuous mapping $D : \mathbb{D}^{1,p}(E) \to L^p(\Omega; H \otimes E)$, called the *derivative operator*.

The operator $D$ is local in the following sense: $1_{\{F=0\}} DF = 0$, for all $F \in \mathbb{D}^{1,2}(E)$. This fact justifies the following definition: A random variable $F : \Omega \to E$ belongs to $\mathbb{D}^{1,2}_{\text{loc}}(E)$ if there exists an increasing sequence $\{\Omega_n\}_n$ of measurable sets converging to $\Omega$ almost surely, and a sequence $\{F_n\}_n \subset \mathbb{D}^{1,2}(E)$ such that $F_n = F$ on $\Omega_n$. For $F \in \mathbb{D}^{1,2}_{\text{loc}}(E)$, the derivative $DF$ is defined as $DF(\omega) = DF_n(\omega)$, if $\omega \in \Omega_n$.

For all $p > 1$, define $\mathbb{L}^{1,p} = L^p([0,1]; \mathbb{D}^{1,p})$. We denote by $\mathbb{L}^{1,p}_C$ the set of processes $u \in \mathbb{L}^{1,p}$ such that:

(1) The set of $L^p(\Omega)$-valued functions $\{s \mapsto D_t u_s, s \in [0,t]\}_{t \in [0,1]}$ is equicontinuous for some version of $Du$, and similarly, the set of functions $\{s \mapsto D_t u_s, s \in [t,1]\}_{t \in [0,1]}$ is also equicontinuous for some (possibly different) version of $Du$.

(2) $\text{ess sup}_{(s,t) \in [0,1]^2} E[|D_t u_s|^p] < \infty$.

The adjoint of the unbounded operator $D : L^2(\Omega) \to L^2(\Omega; H)$ is called the *Skorohod integral* and denoted by $\delta$. It satisfies the following local property: $1_{\{\int_0^1 u_t^2 \mathrm{d}t = 0\}} \delta(u) = 0$, for all $u \in \mathbb{L}^{1,p}$, $p > 1$. We can introduce as before the local spaces $\mathbb{L}^{1,p}_{\text{loc}}$ and $(\mathbb{L}^{1,p}_C)_{\text{loc}}$, for $p > 1$. Then we have (cf. Theorem 7.3 of Nualart and Pardoux, 1988):

**Proposition 2.1.** *If $u \in (\mathbb{L}^{1,2}_C)_{\text{loc}}$, then:*

(1) *The limits $D_s^+ u_s = \lim_{t \downarrow s} D_t u_s$ and $D_s^- u_s = \lim_{t \uparrow s} D_t u_s$ exist in probability.*

(2) *$u$ belongs to the domain of the operator $\delta$ and is Stratonovich integrable.*

(3) $$\int_0^1 u_s \circ \mathrm{d}W_s = \delta(u) + \frac{1}{2} \int_0^1 (D_s^+ u_s + D_s^- u_s)\,\mathrm{d}s.$$

We also need the following different concept of differentiability:

**Definition 2.2.** A mapping $u : \Omega \to H$ is $H - C^1$ if there exists a random kernel $Du(\omega) \in L^2([0,1]^2)$ such that:

(1) $\| u(\omega + \int_0^{\cdot} h_s \mathrm{d}s) - u(\omega) - [Du(\omega)](h) \|_H = o(\| h \|)$, as $\| h \|_H \to 0$, a.s.

(2) The mapping $h \mapsto Du(\omega + \int_0^{\cdot} h_s \mathrm{d}s)$ is continuous from $H$ into $L^2([0,1]^2)$, a.s.

If $u$ is $H - C^1$, then $u \in \mathbb{L}^{1,2}_{\text{loc}}$, and the kernel verifying the above conditions (1) and (2) is precisely the derivative $Du$.

For any Hilbert–Schmidt operator $\mathcal{K}$ on a Hilbert space $H$, its Carleman–Fredholm determinant, denoted by $\det_2(I_H + \mathcal{K})$, is defined by

$$\det_2(I_H + \mathcal{K}) := \prod_{i=1}^{\infty} (1 + \lambda_i) e^{-\lambda_i},$$

where $\{\lambda_i\}_{i=1}^{\infty}$ is the family of eigenvalues of $\mathcal{K}$, countd with their multiplicity. For the properties of this quantity and its role in the theory of integral equations see, for instance, Cochran (1972).

The following is a Girsanov-type theorem for anticipating transformations. It will be our main tool in Section 4.

**Theorem 2.3.** (Ramer, 1974; Kusuoka, 1982). *Let $u:\Omega \to H$ be $H - C^1$. Suppose that:*

(a) *The transformation $T:\Omega \to \Omega$ given by $T(\omega)_t = \omega_t + \int_0^t u_s(\omega)\,\mathrm{d}s$ is bijective.*

(b) *The operator $I + Du: H \to H$ is invertible, a.s.*

*Then, $Q := P \circ T$ (the image of the Wiener measure through $T^{-1}$) is equivalent to $P$ and*

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = |\det_2(I + Du)| \exp\{-\delta(u) - \tfrac{1}{2}\|u\|_H^2\}. \tag{2.1}$$

**Remarks 2.4.** There exist stronger versions of Theorem 2.3, but we will not make use of them. Particularly, Üstünel and Zakai (1993, 1994) have obtained representations for the density of $Q$ without hypothesis (a) and with less regularity on F.

The whole of this section can be stated in the context of abstract Wiener spaces without difficulty. □

## 3. Existence and uniqueness

A formal computation, using the fact that the Stratonovich integral follows the rules of ordinary calculus, yields from (1.2),

$$\begin{aligned} &\dot X_t = \mathrm{e}^{\sigma W_t}\dot X_0 + \int_0^t \mathrm{e}^{\sigma(W_t - W_s)} f(X_s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1, \\ &X_0 = 0,\ X_1 = 1. \end{aligned} \tag{3.1}$$

Consider the following proposition, whose proof is a consequence of the definition of the Stratonovich integral.

**Proposition 3.1.** *Let $\{\phi_t, t \in [0,1]\}$ be a process with $C^1$ paths. Then the process $\{\phi_t \mathrm{e}^{\sigma W_t}, t \in [0,1]\}$ is Stratonovich integrable in each interval $[0,t]$, and*

$$\int_0^t \phi_s \mathrm{e}^{\sigma W_s} \circ \mathrm{d}W_s = \frac{1}{\sigma}\left(\phi_t \mathrm{e}^{\sigma W_t} - \phi_0 - \int_0^t \dot\phi_s \mathrm{e}^{\sigma W_s}\,\mathrm{d}s\right). \tag{3.2}$$

Applying Proposition 3.1 to $\phi_t = \dot X_0 + \int_0^t \mathrm{e}^{-\sigma W_s} f(X_s)\,\mathrm{d}s$, we deduce that if $X$ solves (3.1) then it satisfies (1.2). We remark that the converse implication requires additional assumptions (for instance, $X \in (\mathbb{L}_C^{1,4})_{\mathrm{loc}}$, see Alabert et al., 1995), and we will not discuss this here. So, henceforth, we will work with Eq. (3.1) instead of (1.2).

Let $Y = \{Y_t, t \in [0,1]\}$ be the solution to (3.1) for $f \equiv 0$. Clearly,

$$Y_t = \left(\int_0^1 \mathrm{e}^{\sigma W_s}\mathrm{d}s\right)^{-1} \int_0^t \mathrm{e}^{\sigma W_s}\mathrm{d}s, \quad t \in [0,1]. \tag{3.3}$$

Notice that for all $\omega \in \Omega$, the function $Y.(\omega)$ belongs to the class $\Sigma$ of continuously differentiable functions in the closed interval $[0,1]$, which take the value 0 at $t = 0$ and

1 at $t = 1$, and possess a positive derivative in [0, 1]. Moreover, the mapping $\omega \mapsto Y(\omega)$ is a bijection from $\Omega$ to $\Sigma$, with inverse

$$Y^{-1}(\xi) = \frac{1}{\sigma}\log(\dot{\xi}(t)/\dot{\xi}(0)), \quad \xi \in \Sigma. \tag{3.4}$$

Let $T : \Omega \mapsto \Omega$ be the transformation defined by

$$T_t(\omega) = \omega(t) - \int_0^t \frac{f(Y_s(\omega))}{\sigma \dot{Y}_s(\omega)}\,\mathrm{d}s. \tag{3.5}$$

Then we have:

**Proposition 3.2.** *A process $X = \{X_t, t \in [0, 1]\}$ is a solution of (3.1) with paths in $\Sigma$ if and only if there exists $Z : \Omega \to \Omega$ such that $T(Z(\omega)) = \omega$, and $Z$ is uniquely determined by the relation $Y(Z(\omega)) = X(\omega)$.*

**Proof.** Let $X$ be a process with paths in $\Sigma$ and define $Z(\omega) = Y^{-1}(X(\omega))$. The property $T(Z(\omega)) = \omega$ means

$$Z_t(\omega) = \omega_t + \int_0^t \frac{f(Y_s(Z(\omega)))}{\sigma \dot{Y}_s(Z(\omega))}\,\mathrm{d}s. \tag{3.6}$$

From (3.4) we obtain that (3.6) is equivalent to the equality

$$\log(\dot{X}_t/\dot{X}_0) = \int_0^t \frac{f(X_s)}{\dot{X}_s}\,\mathrm{d}s + \sigma W_t$$

or

$$\mathrm{e}^{-\sigma W_t}\dot{X}_t = \dot{X}_0 \mathrm{e}^{\int_0^t [f(X_s)/\dot{X}_s]\,\mathrm{d}s} \tag{3.7}$$

Clearly, (3.7) is equivalent to (3.1). □

**Corollary 3.3.** *If the transformation $T : \Omega \to \Omega$ defined in (3.5) is bijective, then there exists a unique solution of (3.1) with paths in $\Sigma$.*

We want to find conditions on $f$ in order to ensure that $T$ is a bijective transformation. Notice that, from the definition of $T$, this is equivalent to the existence and uniqueness of a solution $v \in C_0([0, 1])$ of the deterministic integral equation, obtained by putting $v_t = \omega_t - \eta_t$,

$$v_t - \frac{1}{\sigma}\int_0^1 \mathrm{e}^{\sigma(v_u + \eta_u)}\,\mathrm{d}u \int_0^t f\left(\frac{\int_0^s \mathrm{e}^{\sigma(v_u + \eta_u)}\,\mathrm{d}u}{\int_0^1 \mathrm{e}^{\sigma(v_u + \eta_u)}\,\mathrm{d}u}\right)\mathrm{e}^{-\sigma(v_s + \eta_s)}\,\mathrm{d}s = 0, \tag{3.8}$$

for each fixed $\eta \in C_0([0, 1])$. Notice also that it is enough to have $f$ defined only on [0, 1].

To study (3.8), we consider, for each $x > 0$, the equation

$$v_t(x) - \frac{x}{\sigma}\int_0^t f\left(\frac{1}{x}\int_0^s \mathrm{e}^{\sigma(v_u(x) + \eta_u)}\,\mathrm{d}u\right)\mathrm{e}^{-\sigma(v_s(x) + \eta_s)}\,\mathrm{d}s = 0. \tag{3.9}$$

If we prove that (3.9) has a unique solution $v_t(x) \in C_0([0,1])$, for each $x > 0$, it will only remain to show that one and only one $x > 0$ solves

$$x = \int_0^1 e^{\sigma(v_s(x) + \eta_s)}\,ds.$$

We use this procedure to show the following theorem.

**Theorem 3.4.** *Let $f:[0,1] \to \mathbb{R}$ be a nonnegative and nondecreasing $C^1$ function such that $f(0) = 0$. Then, Eq.* (3.1) *has a unique solution $X_t$ which paths in $\Sigma$.*

**Proof.** It will be a consequence of the following two lemmas.

**Lemma 1.** *Suppose that $f:\mathbb{R}^+ \to \mathbb{R}$ is a locally Lipschitz, nonnegative, and bounded function. Then, for all $x > 0$, Eq.* (3.9) *has a unique solution $v(x) \in C_0([0,1])$, for each fixed $\eta \in C_0([0,1])$.*

**Proof.** We prove first the uniqueness. Fix $x > 0$ and $\eta \in C_0([0,1])$. Let $v_t(x)$ and $\bar{v}_t(x)$ be two solutions of (3.9). We will have

$$\begin{aligned}
|v_t(x) - \bar{v}_t(x)| &\leqslant \frac{x}{\sigma}\int_0^t \left| f\left(\frac{1}{x}\int_0^s e^{\sigma(v_u(x) + \eta_u)}\,du\right) e^{-\sigma(v_s(x) + \eta_s)} \right. \\
&\quad \left. - f\left(\frac{1}{x}\int_0^s e^{\sigma(\bar{v}_u(x) + \eta_u)}\,du\right) e^{-\sigma(\bar{v}_s(x) + \eta_s)} \right| ds \\
&\leqslant \frac{x}{\sigma}\int_0^t \left\{ \left| f\left(\frac{1}{x}\int_0^s e^{\sigma(v_u(x) + \eta_u)}\,du\right) - f\left(\frac{1}{x}\int_0^s e^{\sigma(\bar{v}_u(x) + \eta_u)}\,du\right) \right| e^{-\sigma(v_s(x) + \eta_s)} \right. \\
&\quad \left. + f\left(\frac{1}{x}\int_0^s e^{\sigma(\bar{v}_u(x) + \eta_u)}\,du\right) | e^{-\sigma(v_s(x) + \eta_s)} - e^{-\sigma(\bar{v}_s(x) + \eta_s)} | \right\} ds \\
&\leqslant \frac{x}{\sigma} C \int_0^t \left\{ \left|\frac{1}{x}\right| \left| \int_0^s (e^{\sigma v_u(x)} - e^{\sigma \bar{v}_u(x)}) e^{\sigma\eta_u}\,du \right| e^{-\sigma(v_s(x) + \eta_s)} \right. \\
&\quad \left. + | e^{-\sigma v_s(x)} - e^{-\sigma \bar{v}_s(x)} | e^{-\sigma\eta_s} \right\} ds,
\end{aligned} \tag{3.10}$$

from some constant $C$, using the Lipschitz property and that $f$ is bounded. Consequently, we can write, for some constant $C$,

$$|v_t(x) - \bar{v}_t(x)| \leqslant C \int_0^t \left( |v_s(x) - \bar{v}_s(x)| + \int_0^s |v_u(x) - \bar{v}_u(x)|\,du \right) ds,$$

and by a Gronwall-type inequality we deduce $|v_t(x) - \bar{v}_t(x)| = 0$, for all $t \in [0,1]$.

Local existence of a solution in a neighborhood of zero for the Cauchy problem

$$\frac{dv_t(x)}{dt} = \frac{x}{\sigma} f\left(\frac{1}{x}\int_0^t e^{\sigma(v_u(x) + \eta_u)}\,du\right) e^{-\sigma(v_t(x) + \eta_t)},$$

$$v_0(x) = 0$$

can be shown by Picard method, using arguments similar to those of (3.10). We want to see that this local solution can be extended to the whole interval [0, 1]. We denote $\alpha_t = e^{\sigma\eta_t}$. Hypothesis $f \geqslant 0$ implies $\mathrm{d}v_t(x)/\mathrm{d}t \geqslant 0$. On the other hand,

$$\frac{\mathrm{d}v_t(x)}{\mathrm{d}t} \leqslant \frac{x}{\sigma} \sup_{x\in\mathbb{R}^+} f(x) \sup_{t\in[0,1]} \alpha_t^{-1},$$

and this implies that we can extend $v_t(x)$ to [0, 1]. □

**Lemma 2.** *Suppose that $f$ is of class $C^1$ in [0, 1], with $f(0) = 0$, $f \geqslant 0$ and $f' \geqslant 0$. Then, if $v_t(x)$ is the solution of (3.9), the mapping $\Phi: \mathbb{R}^+ - \{0\} \to \mathbb{R}^+ - \{0\}$, defined by*

$$x \mapsto \int_0^1 e^{\sigma(v_u(x) + \eta_u)} \mathrm{d}u$$

*has a unique fixed point.*

**Proof.** Put, as before, $\alpha_t = e^{\sigma\eta_t}$, and denote also $\beta_t(x) = e^{\sigma v_t(x)}/x$, for $x > 0$. Differentiating with respect to $t$, we can write (3.9) as

$$\frac{\sigma}{x} e^{\sigma v_t(x)} \frac{\mathrm{d}v_t(x)}{\mathrm{d}t} = f\left(\frac{1}{x}\int_0^t e^{\sigma(v_u(x)+\eta_u)}\mathrm{d}u\right) e^{-\sigma\eta_t}, \tag{3.11}$$

or, in the notation just introduced,

$$\frac{\mathrm{d}\beta_t(x)}{\mathrm{d}t} = f\left(\int_0^t \alpha_u\beta_u(x)\,\mathrm{d}u\right)\alpha_t^{-1}.$$

Differentiating with respect to $x$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\beta_t(x)}{\mathrm{d}t} = f'\left(\int_0^t \alpha_u\beta_u(x)\,\mathrm{d}u\right)\alpha_t^{-1}\int_0^t \alpha_u \frac{\mathrm{d}\beta_u(x)}{\mathrm{d}x}\,\mathrm{d}u.$$

Denoting $\gamma_t(x) = f'(\int_0^t \alpha_u\beta_u(x)\,\mathrm{d}u)\alpha_t^{-1} \geqslant 0$, and integrating with respect to $t$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\beta_t(x) = \frac{\mathrm{d}}{\mathrm{d}x}\beta_0(x) + \int_0^t \gamma_s(x)\int_0^s \alpha_u \frac{\mathrm{d}}{\mathrm{d}x}\beta_u(x)\,\mathrm{d}u\,\mathrm{d}s. \tag{3.12}$$

But $(\mathrm{d}/\mathrm{d}x)\beta_0(x) = -1/x^2$, together with (3.12), implies that $(\mathrm{d}/\mathrm{d}x)\beta_t(x) < 0$, for all $x$. Therefore, $\beta_t(x)$ is decreasing in $x$ for each fixed $t$, giving that $\Phi(x)/x$ is also decreasing. We deduce that $\Phi$ has at most one fixed point.

We turn to the existence of that fixed point. We know from the proof of Lemma 1 that $v_t(x)$ is nonnegative for all $t$ and $x$. Consequently,

$$\lim_{x\to 0} \Phi(x) \geqslant \int_0^1 \alpha_u\,\mathrm{d}u > 0. \tag{3.13}$$

On the other hand, $\Phi$ is bounded. Indeed, integrating with respect to $t$ in (3.11), we obtain

$$e^{\sigma v_t(x)} = 1 + x \int_0^t f\left(\frac{1}{x}\int_0^s \alpha_u e^{\sigma v_u(x)} du\right)\alpha_s^{-1} ds,$$

and using $f(x) \leqslant Kx$,

$$e^{\sigma v_t(x)} \leqslant 1 + K \int_0^t \alpha_s^{-1} \int_0^s \alpha_u e^{\sigma v_u(x)} du\, ds$$

$$\leqslant 1 + C \int_0^t \int_0^s e^{\sigma v_u(x)} du\, ds \tag{3.14}$$

for some constant $C$. Inequality (3.14) implies that $\Phi$ is bounded. From this fact and (3.13) we arrive of the existence of the fixed point, because $\lim_{x\to 0} \Phi(x)/x = \infty$ and $\lim_{x\to\infty} \Phi(x)/x = 0$.

The proof of Lemma 2 is complete and Theorem 3.4 holds. □

**Remarks 3.5.** Condition $f(0) = 0$ cannot be removed. Suppose, for instance, that $f \equiv K > 0$. In this case, one can prove that $T$ *is not* bijective.

This does not imply that (3.1) fails to possess a solution. If $f \equiv K \neq 0$, we can solve (3.1) directly, and the solution is

$$X_t = \dot{X}_0 \int_0^t e^{\sigma W_s} ds + K \int_0^t e^{\sigma W_s} \int_0^s e^{-\sigma W_u} du\, ds,$$

where

$$\dot{X}_0 = \left(1 - K \int_0^1 e^{\sigma W_s} \int_0^s e^{-\sigma W_u} du\, ds\right)\left(\int_0^1 e^{\sigma W_s} ds\right)^{-1}.$$

In this case, the paths of the process $\{X_t, t \in [0, 1]\}$ do not belong to $\Sigma$. □

## 4. Markov field property

The idea of the change of measure method to study nonlinear anticipating stochastic differential equations is analogous to that of the classical Girsanov theorem for non-anticipating ones. We have chosen a transformation $T: \Omega \to \Omega$ such that $X_t = T^{-1}(Y)_t$, where $X$ is the solution to our Eq. (3.1) and $Y$, defined in (3.3), solves (3.1) for $f \equiv 0$. Therefore, defining $Q = P \circ T$, the law of $Y_t$ under $Q$ coincides with the law of $X_t$ under $P$. Anything we can prove concerning the first produces automatically the same result for the latter. In other words, we switch to the simpler process $Y$, given in explicit form, at the price of having to deal with a more complicated measure.

First, we will prove that the process $(Y_t, \dot{Y}_t)$ is a Markov process under the original probability $P$.

**Proposition 4.1.** *The two-dimensional process* $\{(Y_t, \dot{Y}_t),\ t \in [0,1]\}$ *given by* (3.3) *is a Markov process under the Wiener measure P.*

**Proof.** It is enough to show that for any $t, s \in [0,1]$, $s > t$, and for any measurable and bounded function $\psi : \mathbb{R}^2 \to \mathbb{R}$, the conditional expectation

$$E_P[\psi(Y_s, \dot{Y}_s)/(Y_r, \dot{Y}_r),\ r \leqslant t] \tag{4.1}$$

is a $(Y_t, \dot{Y}_t)$-measurable random variable. It is easy to see that the $\sigma$-field generated by $\{(Y_r, \dot{Y}_r),\ r \leqslant t\}$ coincides with the one generated by

$$\int_t^1 e^{\sigma(W_r - W_t)}\,dr \quad \text{and} \quad \{W_r, r \leqslant t\}.$$

Denote

$$\alpha = \int_0^1 e^{\sigma W_r}\,dr, \qquad \beta = \int_0^t e^{\sigma W_r}\,dr, \qquad \gamma = e^{\sigma W_t}.$$

$\alpha$, $\beta$ and $\gamma$ are measurable with respect to the conditioning $\sigma$-field, so that we can write (4.1) as

$$E_P\left[\psi\left(\frac{1}{\alpha}\left(\beta + \gamma\int_t^s e^{\sigma(W_r - W_t)}\,dr\right), \frac{1}{\alpha}\gamma e^{\sigma(W_s - W_t)}\right)\Big/ \int_t^1 e^{\sigma(W_r - W_t)}\,dr,\ \{W_r, r \leqslant t\}\right] \tag{4.2}$$

in the sense that $\alpha$, $\beta$ and $\gamma$ are constants that should be given their values after applying the conditional expectation operator.

Notice that the variables $\{W_r, r \leqslant t\}$ are independent of all other variables involved, and therefore they can be removed from (4.2). The resulting expression is a function of

$$\frac{\beta}{\alpha} = \frac{\int_0^t e^{\sigma W_r}\,dr}{\int_0^1 e^{\sigma W_r}\,dr} = Y_t, \qquad \frac{\gamma}{\alpha} = \frac{e^{\sigma W_t}}{\int_0^1 e^{\sigma W_r}\,dr} = \dot{Y}_t,$$

$$\int_t^1 e^{\sigma(W_r - W_t)}\,dr = \frac{1 - Y_t}{\dot{Y}_t},$$

which are $(Y_t, \dot{Y}_t)$-measurable random variables. □

We want to apply Ramer–Kusuoka Theorem to the transformation $T$ defined in (3.5). First, notice that if $f$ is of class $C^1$, then the stochastic process

$$u_s = \frac{-f(Y_s)}{\sigma \dot{Y}_s} \tag{4.3}$$

is $H - C^1$. Its derivative $D_t u_s$ is given by

$$D_t u_s = \begin{cases} \dfrac{Y_t}{\dot{Y}_s}[f(Y_s) + f'(Y_s)(1 - Y_s)] & \text{if } t \leqslant s, \\[2ex] \dfrac{1 - Y_t}{\dot{Y}_s}[-f(Y_s) + f'(Y_s)Y_s] & \text{if } t > s. \end{cases} \tag{4.4}$$

Indeed, using the chain rule for the derivative operator, we have

$$D_t u_s = \frac{-\dot{Y}_s f'(Y_s) D_t Y_s + f(Y_s) D_t \dot{Y}_s}{\sigma \dot{Y}_s^2},$$

with

$$D_t Y_s = \frac{\int_0^1 e^{\sigma W_r} dr \int_0^s \sigma e^{\sigma W_r} \mathbf{1}_{[0,r]}(t)\, dr - \int_0^s e^{\sigma W_r} dr \int_0^1 \sigma e^{\sigma W_r} \mathbf{1}_{[0,r]}(t)\, dr}{(\int_0^1 e^{\sigma W_r} dr)^2}$$

$$= \sigma[-Y_t(1 - Y_s)\mathbf{1}_{[0,s]}(t) - Y_s(1 - Y_t)\mathbf{1}_{]s,1]}(t)]$$

and

$$D_t \dot{Y}_s = \frac{\sigma e^{\sigma W_s} \int_0^1 e^{\sigma W_r} dr\, \mathbf{1}_{[0,s]}(t) - e^{\sigma W_s} \int_0^1 \sigma e^{\sigma W_r} \mathbf{1}_{[0,r]}(t)\, dr\zeta}{(\int_0^1 e^{\sigma W_r} dr)^2}$$

$$= \sigma[\dot{Y}_s Y_t \mathbf{1}_{[0,s]}(t) - \dot{Y}_s(1 - Y_t)\mathbf{1}_{]s,1]}(t)],$$

and we obtain (4.4).

**Proposition 4.2.** *Let $f:[0,1] \to \mathbb{R}$ be a nonnegative, nondecreasing $C^1$ function such that $f(0) = 0$. Let $u$ be the stochastic process defined in* (4.3). *Then,*
(1) *The transformation $T: \Omega \to \Omega$ given by $T(\omega)_t = \omega_t + \int_0^t u_s(\omega)\, ds$ is bijective.*
(2) *The operator $I + Du(\omega)$ is invertible, a.s.*

**Proof.** The bijectivity of $T$ is contained in the proof of Theorem 3.4. It only remains to show the invertibility of the operator in (2).

Suppose $T^{-1}$ is Fréchet differentiable in $\Omega$. Then, the following lemma (see Nualart, 1993) applies.

**Lemma.** *Let $T: \Omega \to \Omega$ be a bijective transformation of the form $T(\omega) = \omega + \int_0^{\cdot} u_s(\omega)\, ds$. (This means that $T^{-1}(\omega) = \omega + \int_0^{\cdot} \bar{u}_s(\omega)\, ds$, with $\bar{u}_s(\omega) = -u_s(T^{-1}(\omega))$.) Suppose that $u: \Omega \to H$ and $T^{-1}: \Omega \to \Omega$ are Fréchet differentiable.*
*Then:*

$$I = [I + (Du)(T^{-1}(\omega))] \circ [I + D\bar{u}(\omega)] \tag{4.5}$$

By the Fredholm alternative, equality (4.5), which can be written

$$I = [I + Du(\omega)] \circ [I + (D\bar{u})(T(\omega))],$$

implies that $I + Du(\omega)$ is invertible and

$$[I + Du(\omega)]^{-1} = I + (D\bar{u})(T(\omega)) = I - (D(u \circ T^{-1}))(T(\omega)).$$

Let us show that $T^{-1}$ is indeed differentiable, and the proof will be complete. Recall that

$$T^{-1}(\omega)_t = Y^{-1}(X(\omega)) = \frac{1}{\sigma} \log(\dot{X}_t(\omega)/\dot{X}_0(\omega)), \tag{4.6}$$

where $X$ is the solution to the system (3.1). Consider the function $F: C([0,1]) \to C([0,1]) \times \mathbb{R}$ given by

$$F(\dot{X}) = \left( \dot{X}_t - e^{\sigma W_t}\left( \dot{X}_0 + \int_0^t e^{-\sigma W_s} f(X_s)\,ds \right), \int_0^1 \dot{X}_t\,dt - 1 \right).$$

It is enough to check that the Fréchet differential $DF(\dot{X})$ is invertible for every $\dot{X} \in C[0,1])$ and the conclusion follows from the Implicit Function Theorem. The derivative of $F$ in the direction of $y \in C([0,1])$ is

$$[DF(\dot{X})](y) = \left( y_t - A_t y_0 - B_t(y), \int_0^1 y_t\,dt \right),$$

where

$$A_t := e^{\sigma W_t},$$

$$B_t(y) := \int_0^t e^{\sigma(W_t - W_s)} f'(X_s) \int_0^s y_u\,du\,ds.$$

Fix $(h, \alpha) \in C([0,1]) \times \mathbb{R}$. The equation $y_t - A_t y_0 - B_t(y) = h_t$ has a unique solution $y(y_0)$ for each $y_0 \in \mathbb{R}$, and it only remains to show that there exists a unique $y_0$ such that

$$\left( \int_0^1 A_t\,dt \right) y_0 + \int_0^1 B_t(y(y_0))\,dt = \alpha. \tag{4.7}$$

We have

$$\begin{aligned} \frac{d}{dy_0} y_t(y_0) &= A_t + \frac{d}{dy_0} B_t(y(y_0)) \\ &= A_t + \int_0^t e^{\sigma(W_t - W_s)} f'(X_s) \int_0^s \frac{d}{dy_0} y_u(y_0)\,du\,ds, \end{aligned}$$

which, together with the initial condition $(d/dy_0)y_t(y_0)|_{t=0} = 1$, yields $(d/dy_0)y_t(y_0) \geqslant A_t$, $\forall t \in [0,1]$. Therefore, $(d/dy_0)B_t(y(y_0)) \geqslant 0 \geqslant -A_t$, and this implies clearly that (4.7) has a unique solution $y_0$. □

Therefore, under the hypotheses of Proposition 4.2, the Ramer–Kusuoka Theorem can be applied and we have the equivalence of the probabilities $P$ and $Q$, with $dQ/dP$ given by formula (2.1). We are going now to compute the Carleman–Fredholm determinant that appears in this density.

**Proposition 4.3.** *Let $Du$ be given by (4.4). Then,*

$$\det_2(I + D_t u_s) = \left( 1 - \int_0^1 g(s)\,ds \right) \exp\left\{ \int_0^1 \frac{1 - Y_s}{\dot{Y}_s} [f(Y_s) - f'(Y_s) Y_s]\,ds \right\}, \tag{4.8}$$

*where $g: [0,1] \to \mathbb{R}$ is the solution of the Volterra equation*

$$g(t) + \int_0^t M(t,s) g(s)\,ds = M(t,0), \tag{4.9}$$

with

$$M(t,s) = \frac{1-Y_t}{\dot{Y}_t}[f(Y_t) - f'(Y_t)(Y_t - Y_s)]\frac{1}{1-Y_s}. \tag{4.10}$$

**Proof.** We will denote

$$\varphi(t) := Y_t \qquad \psi(t) := \frac{1}{\dot{Y}_t}[f(Y_t) + f'(Y_t)(1-Y_t)],$$

$$\theta(t) := \frac{1}{\dot{Y}_t}[-f(Y_t) + f'(Y_t)Y_t]. \tag{4.11}$$

$D_t u_s$ can then be expressed

$$D_t u_s = \varphi(t)\psi(s)\mathbf{1}_{\{t \leq s\}} + (1-\varphi(t))\theta(s)\mathbf{1}_{\{t>s\}}.$$

For each fixed $n \in \mathbb{N}$, denote $t_i = i/n$ and define $e_i(t) := \sqrt{n} \cdot \mathbf{1}_{]t_{i-1}, t_i]}(t)$ for $i = 1, \dots, n$. The function

$$K^{(n)}(t,s) = \frac{1}{n}\sum_{i,j=1}^{n}[\varphi(t_{i-1})\psi(t_{j-1})\mathbf{1}_{\{i \leq j\}} + (1-\varphi(t_{i-1}))\theta(t_{j-1})\mathbf{1}_{\{i>j\}}]$$
$$\times e_i(t)e_j(s) \tag{4.12}$$

converges to $D_t u_s$ in $L^2([0,1]^2)$ as $n$ increases.

We can compute $\det_2(I + D_t u_s)$ as $\lim_{n\to\infty} \det_2(I + K^{(n)}(t,s))$, since the Carleman–Fredholm determinant is continuous in $L^2([0,1]^2)$. On the other hand, the matrix $B^{(n)} = (b_{ij}^{(n)})$, where

$$b_{ij}^{(n)} = \begin{cases} \frac{1}{n}\varphi(t_{i-1})\psi(t_{j-1}) & \text{if } i \leq j, \\ \frac{1}{n}(1-\varphi(t_{i-1}))\theta(t_{j-1}) & \text{if } i > j \end{cases},$$

satisfies

$$\det_2(I + K^{(n)}(t,s)) = \det(I_{\mathbb{R}^n} + B^{(n)})\exp\{-\operatorname{tr} B^{(n)}\}.$$

The trace of $-B^{(n)}$ converges as $n \to \infty$ to

$$-\int_0^1 \varphi(t)\psi(t)\,dt. \tag{4.13}$$

Concerning the determinant,

$$\det(I_{\mathbb{R}^n} + B^{(n)}) = \frac{1}{n^n}\times$$

$$\times \begin{vmatrix} n+\varphi(t_0)\psi(t_0) & \varphi(t_0)\psi(t_1) & \cdots & & \\ (1-\varphi(t_1))\theta(t_0) & n+\varphi(t_1)\psi(t_1) & \varphi(t_1)\psi(t_2) & \cdots & \\ (1-\varphi(t_2))\theta(t_0) & (1-\varphi(t_2))\theta(t_1) & n+\varphi(t_2)\psi(t_2) & & \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \varphi(t_{n-2})\psi(t_{n-1}) \\ (1-\varphi(t_{n-1}))\theta(t_0) & \cdots & \cdots & \cdots & n+\varphi(t_{n-1})\psi(t_{n-1}) \end{vmatrix} \tag{4.14}$$

Since $\varphi(t_0) = 0$, we suppress the first row and the first column. Subtracting the $(i+1)$th row from the $i$th row $(i = 1, \dots, n-2)$ and then subtracting the $(i+1)$th row from the $i$th one multiplied by

$$\frac{\varphi(t_{i+1}) - \varphi(t_i)}{\varphi(t_{i+2}) - \varphi(t_{i+1})} \quad (i = 1, \dots, n-2),$$

we arrive at

$$\det(I_{\mathbb{R}^n} + B^{(n)}) = \frac{1}{n^{n-1}} \cdot \begin{vmatrix} a_1^{(n)} & b_1^{(n)} & c_1^{(n)} & 0 & \cdots & & \\ 0 & a_2^{(n)} & b_2^{(n)} & c_2^{(n)} & 0 & \cdots & \\ \vdots & 0 & a_3^{(n)} & b_3^{(n)} & \ddots & & \\ & \vdots & & \ddots & \ddots & & \\ & & & & & b_{n-3}^{(n)} & c_{n-3}^{(n)} \\ & & & & & a_{n-2}^{(n)} & b_{n-2}^{(n)} \\ d_1^{(n)} & d_2^{(n)} & d_3^{(n)} & \cdots & \cdots & d_{n-2}^{(n)} & e^{(n)} \end{vmatrix} \tag{4.15}$$

with

$$a_i^{(n)} = n + \varphi(t_i)\psi(t_i) - (1 - \varphi(t_i))\theta(t_i), \quad i = 1, \dots, n-2,$$

$$b_i^{(n)} = -n - \frac{\varphi(t_{i+1}) - \varphi(t_i)}{\varphi(t_{i+2}) - \varphi(t_{i+1})}(\varphi(t_{i+2})\psi(t_{i+1}) + n - (1 - \varphi(t_{i+2}))\theta(t_{i+1})),$$

$i = 1, \dots, n-3$,

$$b_{n-2}^{(n)} = -n - \frac{\varphi(t_{n-1}) - \varphi(t_{n-2})}{\varphi(t_n) - \varphi(t_{n-1})}(\varphi(t_n)\psi(t_{n-1}) + n), \quad c_i^{(n)} = n\frac{\varphi(t_{i+1}) - \varphi(t_i)}{\varphi(t_{i+2}) - \varphi(t_{i+1})},$$

$i = 1, \dots, n-3$,

$$d_i^{(n)} = (\varphi(t_n) - \varphi(t_{n-1}))\theta(t_i),$$

$$e^{(n)} = n + \varphi(t_{n-1})\psi(t_{n-1}).$$

We transform (4.15) into triangular form by adding to the last row a suitable combination of the others. Let $m_i^{(n)}$ be the multiplier of the $i$th row:

$$a_1^{(n)}m_1^{(n)} + d_1^{(n)} = 0,$$

$$b_1^{(n)}m_1^{(n)} + a_2^{(n)}m_2^{(n)} + d_2^{(n)} = 0,$$

$$c_i^{(n)}m_i^{(n)} + b_{i+1}^{(n)}m_{i+1}^{(n)} + a_{i+2}^{(n)}m_{i+2}^{(n)} + d_{i+2}^{(n)} = 0, \quad i = 1, \dots, n-4. \tag{4.16}$$

The determinant in (4.15) can be expressed:

$$\det(I_{\mathbb{R}^n} + B^{(n)}) = \frac{1}{n^{n-1}}(e^{(n)} + b_{n-2}^{(n)}m_{n-2}^{(n)} + c_{n-3}^{(n)}m_{n-3}^{(n)}) \prod_{i=1}^{n-2} a_i^{(n)}.$$

On the one hand,

$$\begin{aligned}
\lim_{n\to\infty} \prod_{i=1}^{n-2} \frac{a_i^{(n)}}{n} &= \exp \lim_{n\to\infty} \sum_{i=1}^{n-2} \log \frac{a_i^{(n)}}{n} \\
&= \exp \lim_{n\to\infty} \sum_{i=1}^{n-2} \log\left(1 + \frac{\varphi(t_i)\psi(t_i) - (1-\varphi(t_i))\theta(t_i)}{n}\right) \\
&= \exp \lim_{n\to\infty} \sum_{i=1}^{n-2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{\varphi(t_i)\psi(t_i) - (1-\varphi(t_i))\theta(t_i)}{n}\right)^k,
\end{aligned} \tag{4.17}$$

where the last equality is valid for $n$ large enough. Given that $\varphi$, $\psi$ and $\theta$ are continuous, the double series converges absolutely and we can apply Fubini and Dominated Convergence Theorems, obtaining that (4.17) is equal to

$$\exp\left\{\int_0^1 [\varphi(t)\psi(t) - (1-\varphi(t))\theta(t)]\,dt\right\}. \tag{4.18}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}(e^{(n)} + b_{n-2}^{(n)}m_{n-2}^{(n)} + c_{n-2}^{(n)}m_{n-3}^{(n)}) \\
&= \lim_{n\to\infty} \frac{1}{n}\Bigg[n + \varphi(t_{n-1})\psi(t_{n-1}) \\
&\quad + \left(-n - \frac{\varphi(t_{n-1}) - \varphi(t_{n-2})}{\varphi(t_n) - \varphi(t_{n-1})}(\varphi(t_n)\psi(t_{n-1}) + n)\right)m_{n-2}^{(n)} \\
&\quad + n\frac{\varphi(t_{n-2}) - \varphi(t_{n-3})}{\varphi(t_{n-1}) - \varphi(t_{n-2})}m_{n-3}^{(n)}\Bigg] \\
&= 1 - 2\lim_{n\to\infty} m_{n-2}^{(n)} + \lim_{n\to\infty} m_{n-3}^{(n)},
\end{aligned} \tag{4.19}$$

taking into account that $\varphi$ is differentiable, and provided the last two limits exist. It only remains to see that these limits exist and to compute them.

Multiplying by $n$ in (4.16), we can write this equality, for $i = 3, \dots, n-4$, as

$$\begin{aligned} n^2\Bigg[&\frac{\varphi(t_{i+1})-\varphi(t_i)}{\varphi(t_{i+2})-\varphi(t_{i+1})}\, m_i^{(n)} \\ &+\left(-1-\frac{\varphi(t_{i+2})-\varphi(t_{i+1})}{\varphi(t_{i+3})-\varphi(t_{i+2})}\right)m_{i+1}^{(n)} + m_{i+2}^{(n)}\Bigg] \\ &+ n\Bigg[(\varphi(t_{i+2})\psi(t_{i+2}) - (1-\varphi(t_{i+2}))\theta(t_{i+2}))m_{i+2}^{(n)} \\ &- \frac{\varphi(t_{i+2})-\varphi(t_{i+1})}{\varphi(t_{i+3})-\varphi(t_{i+2})}(\varphi(t_{i+3})\psi(t_{i+2}) - (1-\varphi(t_{i+3}))\theta(t_{i+2})m_{i+1}^{(n)}\Bigg] \\ &+ n[\varphi(t_n)-\varphi(t_{n-1})]\theta(t_{i+2}) = 0. \end{aligned} \tag{4.20}$$

Define

$$p_i^n := n^2[\xi_i m_i^{(n)} + (-1-\xi_{i+1})m_{i+1}^{(n)} + m_{i+2}^{(n)}] \quad \text{for } i = 1, 2, \dots, n-4,$$

with

$$\xi_i := \frac{\varphi(t_{i+1})-\varphi(t_i)}{\varphi(t_{i+2})-\varphi(t_{i+1})}. \tag{4.21}$$

Then,

$$\begin{aligned} \sum_{k=1}^{i} p_k^{(n)} &= n^2[\xi_1 m_1^{(n)} + (-1-\xi_2)m_2^{(n)} + m_3^{(n)} \\ &\quad + \xi_2 m_2^{(n)} + (-1-\xi_3)m_3^{(n)} + m_4^{(n)} \\ &\quad + \xi_{i-1} m_{i-1}^{(n)} + (-1-\xi_i)m_i^{(n)} + m_{i+1}^{(n)} \\ &\quad + \xi_i m_i^{(n)} + (-1-\xi_{i+1})m_{i+1}^{(n)} + m_{i+2}^{(n)}] \\ &= n^2[\xi_1 m_1^{(n)} - m_2^{(n)} - \xi_{i+1}m_{i+1}^{(n)} + m_{i+2}^{(n)}], \end{aligned} \tag{4.22}$$

and we also have

$$\begin{aligned} &\sum_{k=1}^{i}(\varphi(t_{i+3})-\varphi(t_{k+2}))\sum_{l=1}^{k} p_l^{(n)} \\ &= \sum_{k=1}^{i} n^2[(\varphi(t_{k+3})-\varphi(t_{k+2}))(\xi_1 m_i^{(n)} - m_2^{(n)}) - (\varphi(t_{k+2})-\varphi(t_{k+1}))m_{k+1}^{(n)} \\ &\quad + (\varphi(t_{k+3})-\varphi(t_{k+2}))m_{k+2}^{(n)}] \\ &= n^2(\varphi(t_{i+3})-\varphi(t_3))(\xi_1 m_1^{(n)} - m_2^{(n)}) \\ &\quad + n^2((\varphi(t_{i+3})-\varphi(t_{i+2}))m_{i+2}^{(n)} - (\varphi(t_3)-\varphi(t_2))m_2^{(n)}) \\ &= n^2((\varphi(t_{i+3})-\varphi(t_3))\xi_1 m_1^{(n)} - (\varphi(t_{i+3})-\varphi(t_2))m_2^{(n)} + (\varphi(t_{i+3})-\varphi(t_{i+2}))m_{i+2}^{(n)}). \end{aligned}$$

Solving for $m_{i+2}^{(n)}$:

$$m_{i+2}^{(n)} = \left(\eta_{i+2} + \frac{1}{n^2}\sum_{k=1}^{i}(\varphi(t_{k+3}) - \varphi(t_{k+2}))\sum_{l=1}^{k} p_l^{(n)}\right)\frac{1}{\varphi(t_{i+3}) - \varphi(t_{i+2})}, \tag{4.23}$$

where

$$\eta_i := -(\varphi(t_{i+1}) - \varphi(t_3))\xi_1 m_1^{(n)} + (\varphi(t_{i+1}) - \varphi(t_2))m_2^{(n)}. \tag{4.24}$$

Substituting into equality (4.20),

$$\begin{aligned} 0 = p_i^{(n)} &+ n[(\varphi(t_{i+2})\psi(t_{i+2}) - (1 - \varphi(t_{i+2}))\theta(t_{i+2})) \\ &\times\left(\eta_{i+2}^{(n)} + \frac{1}{n^2}\sum_{k=1}^{i}(\varphi(t_{k+3}) - \varphi(t_{k+2}))\sum_{l=1}^{k} p_l^{(n)}\right)\frac{1}{\varphi(t_{i+3}) - \varphi(t_{i+2})} \\ &- \frac{\varphi(t_{i+2}) - \varphi(t_{i+1})}{\varphi(t_{i+3}) - \varphi(t_{i+2})}(\varphi(t_{i+3})\psi(t_{i+2}) - (1 - \varphi(t_{i+3}))\theta(t_{i+2})) \\ &\times\left(\eta_{i+1}^{(n)} + \frac{1}{n^2}\sum_{k=1}^{i-1}(\varphi(t_{k+3}) - \varphi(t_{k+2}))\sum_{l=1}^{k} p_l^{(n)}\right)\frac{1}{\varphi(t_{i+2}) - \varphi(t_{i+1})} \\ &+ n[\varphi(t_n) - \varphi(t_{n-1})]\theta(t_{i+2}). \end{aligned} \tag{4.25}$$

The last term tends to $\varphi'(1)\cdot\theta(t)$ as $n \to \infty$. If we multiply and divide by $n$, the second term can be written

$$\begin{aligned} &\Bigg[(n(\varphi(t_{i+2}) - \varphi(t_{i+3}))(\psi(t_{i+2}) + \theta(t_{i+2})) \\ &\quad\times\left(n\eta_{i+2}^{(n)} + \frac{1}{n}\sum_{k=1}^{i} n(\varphi(t_{k+3}) - \varphi(t_{k+2}))\frac{1}{n}\sum_{l=1}^{k} p_l^{(n)}\right) \\ &\quad + (\varphi(t_{i+3})\psi(t_{i+2}) - (1 - \varphi(t_{i+3}))\theta(t_{i+2})) \\ &\quad\times\left(n^2(\eta_{i+2}^{(n)} - \eta_{i+1}^{(n)}) + n(\varphi(t_{i+3}) - \varphi(t_{i+2}))\frac{1}{n}\sum_{l=1}^{i} p_l^{(n)}\right)\Bigg] \\ &\quad\times\frac{1}{n(\varphi(t_{i+3}) - \varphi(t_{i+2}))}. \end{aligned}$$

It is immediate that

$$\lim_{n\to\infty} n(\varphi(t_{i+2}) - \varphi(t_{i+3}))(\psi(t_{i+2}) + \theta(t_{i+2})) = -\varphi'(t)(\psi(t) + \theta(t)),$$

$$\lim_{n\to\infty} \varphi(t_{i+3})\psi(t_{i+2}) - (1 - \varphi(t_{i+3}))\theta(t_{i+2}) = \varphi(t)(\psi(t) - (1 - \varphi(t))\theta(t).$$

Let us see that $n\eta_{i+2}^{(n)}$ and $n^2(\eta_{i+2}^{(n)} - \eta_{i+1}^{(n)})$ both have limit zero: Recalling the definitions of $m_1^{(n)}$ and $m_2^{(n)}$ (4.16), $\xi_1$ (4.21), and $\eta_i^{(n)}$ (4.24), we have

$$\begin{aligned} n\eta_{i+2}^{(n)} &= (\varphi(t_{i+3}) - \varphi(t_3))\frac{\varphi(t_2) - \varphi(t_1)}{\varphi(t_3) - \varphi(t_2)}\cdot n\frac{d_1^{(n)}}{a_1^{(n)}} \\ &\quad - (\varphi(t_{i+3}) - \varphi(t_2))\cdot n\frac{d_2^{(n)} + b_1^{(n)}m_1^{(n)}}{a_2^{(n)}}. \end{aligned} \tag{4.26}$$

The first term has limit zero, because $\lim_{n\to\infty} nd_1^{(n)} = \varphi'(1)\theta(0)$, $\lim_{n\to\infty} a_1^{(n)} = +\infty$, and the other factor tends to $\varphi(t)$. Similarly, $\lim_{n\to\infty} nd_2^{(n)}/a_2^{(n)} = 0$, and

$$\lim_{n\to\infty} \frac{nb_1^{(n)}m_1^{(n)}}{a_2^{(n)}}$$

$$= \lim_{n\to\infty} \frac{\left(n + \frac{\varphi(t_2)-\varphi(t_1)}{\varphi(t_3)-\varphi(t_2)}(n + \varphi(t_{i+2})\psi(t_{i+2}) - (1-\varphi(t_{i+2}))\theta(t_{i+1}))\right)\cdot nd_1^{(n)}}{a_1^{(n)}a_2^{(n)}} = 0,$$

because the order of numerator and denominator is $n$ and $n^2$, respectively, for large $n$.

On the other hand, from (4.24),

$$n^2(\eta_{i+2}^{(n)} - \eta_{i+1}^{(n)}) = n(\varphi(t_{i+3}) - \varphi(t_{i+2}))(nm_2^{(n)} - n\xi_1 m_1^{(n)}).$$

Both terms in the second factor appear in (4.26) and we have just shown that they tend to zero, while the first factor tends to $\varphi'(t)$.

The jump function defined by $p_i^{(n)}$, i.e.

$$z^{(n)}(t) := \sum_i p_i^{(n)} \cdot \mathbf{1}_{]t_i, t_{i+1}]}(t),$$

converges a.e. and boundedly on $[0,1]$ to the solution $z$ of the Volterra integral equation

$$z(t) + \int_0^t [\psi(t)\varphi(s) - (1-\varphi(s))\theta(t)]z(s)\,ds = -\varphi'(1)\theta(t).$$

To see this, notice first that, in view of (4.25) and the limits above, one can write

$$|z^{(n)}(t)| \leqslant A^{(n)} + B^{(n)}\int_0^t |z^{(n)}(s)|\,ds + C^{(n)}\int_0^t\int_0^s |z^{(n)}(u)|\,du\,ds \tag{4.27}$$

for some converging sequences $A^{(n)}, B^{(n)}, C^{(n)}$, whence $|z^{(n)}(t)|$ is uniformly bounded in $n$ and $t$. Taking this into account, an inequality similar to (4.27) is obtained for $|z^{(n)}(t) - z(t)|$, $t \in ]0, 1-3/n]$. We will have then $\lim_{n\to\infty} z^{(n)} = z$, a.e. and boundedly on $[0,1]$.

We can now turn back to (4.23), to compute the limit we are interested in:

$$\lim_{n\to\infty} m_{n-3}^{(n)} = \lim_{n\to\infty}\left(\eta_{n-3}^{(n)} + \frac{1}{n^2}\sum_{k=1}^{n-5}(\varphi(t_{k+3}) - \varphi(t_{k+2}))\sum_{l=1}^{k} p_l^{(n)}\right)\frac{1}{\varphi(t_{n-2}) - \varphi(t_{n-3})}$$

$$= \frac{1}{\varphi'(1)}\left(\lim_{n\to\infty} n\eta_{n-3}^{(n)} + \frac{1}{n}\sum_{k=1}^{n-5} n(\varphi(t_{k+3}) - \varphi(t_{k+2}))\frac{1}{n}\sum_{l=1}^{k} p_l^{(n)}\right)$$

$$= \frac{1}{\varphi'(1)}\int_0^1 \varphi'(s)\int_0^s z(u)\,du = \frac{1}{\varphi'(1)}\int_0^1 z(s)\int_s^1 \varphi'(u)\,du$$

$$= \int_0^1 \frac{1-\varphi(s)}{\varphi'(1)} z(s)\,ds.$$

The computations for $\lim_{n\to\infty} m^{(n)}_{n-2}$ are the same, with the same result, and therefore, using (4.19), (4.18) and (4.13), we find that

$$\det{}_2(I + D_t u_s) = \left(1 - \int_0^1 \frac{1-\varphi(s)}{\varphi'(1)} z(s)\,ds\right)$$
$$\times \exp\left\{-\int_0^1 \varphi(t)\psi(t)\,dt\right\} \exp\left\{\int_0^1 [\psi(t)\varphi(t) - (1-\varphi(t))\theta(t)]\,dt\right\}$$
$$= \left(1 - \int_0^1 \frac{1-\varphi(s)}{\varphi'(1)} z(s)\,ds\right) \exp\left\{-\int_0^1 (1-\varphi(t))\theta(t)\,dt\right\},$$

Alternatively, if we define

$$g(s) = \frac{1-\varphi(s)}{\varphi'(1)} z(s),$$

we obtain

$$\det{}_2(I + D_t u_s) = \left(1 - \int_0^1 g(s)\,ds\right) \exp\left\{-\int_0^1 (1-\varphi(t))\theta(t)\,dt\right\},$$

with $g(t)$ the solution of the integral equation

$$g(t) + \int_0^t M(t,s) g(s)\,ds = M(t,0),$$

where

$$M(t,s) := \frac{1-\varphi(t)}{1-\varphi(s)} [\psi(t)\varphi(s) - (1-\varphi(s))\theta(t)].$$

Using now the notations (4.11), we find (4.8). □

It is easy to show that, under the hypotheses of the previous proposition, the process $u_s = -f(Y_s)/\sigma \dot{Y}_s$ belongs to $\mathbb{L}_C^{1,2}$ and we can apply Proposition 2.1 to obtain

$$\int_0^1 u_s \circ dW_s = \delta(u) + \int_0^1 \left(\frac{1-Y_s}{\dot{Y}_s}[-f(Y_s) + f'(Y_s)Y_s] + \frac{f(Y_s)}{2\dot{Y}_s}\right) ds.$$

Formula (2.1) can be written (using (4.8)) as

$$\frac{dQ}{dP} = \left|1 - \int_0^1 g(s)\,ds\right|$$
$$\times \exp\left\{\int_0^1 \frac{f(Y_s)}{\sigma \dot{Y}_s} \circ dW_s + \frac{1}{2}\int_0^1 \left[\frac{f(Y_s)}{\dot{Y}_s} - \left(\frac{f(Y_s)}{\sigma \dot{Y}_s}\right)^2\right] ds\right\},$$

where $g$ is given by (4.9).

Fix two points $0 \leqslant s < t \leqslant 1$, and denote

$$\mathfrak{F}^{\mathrm{i}} := \sigma(Y_u, \dot{Y}_u), u \in [s,t]\}, \qquad \mathfrak{F}^{\mathrm{e}} := \sigma\{(Y_u, \dot{Y}_u), u \in ]s,t[^{\mathrm{c}}\},$$
$$\mathfrak{F}^{\mathrm{b}} := \sigma\{Y_s, \dot{Y}_s, Y_t, \dot{Y}_t\}. \tag{4.28}$$

We know that $(X_t, \dot{X}_t)$ is a Markov field under $P$ iff $(Y_t, \dot{Y}_t)$ is a Markov field under $Q$, and this is in turn equivalent to say that for any bounded and $\mathfrak{F}^e$-measurable random variable $\xi$, $E[\xi/\mathfrak{F}^i]$ is an $\mathfrak{F}^b$-measurable random variable, for all possible choices of $s$ and $t$. It is easy to show that

$$E_Q[\xi/\mathfrak{F}^i] = \frac{E_P[\xi/\mathfrak{F}^i]}{E_P[J/\mathfrak{F}^i]}, \tag{4.29}$$

where $J := \mathrm{d}Q/\mathrm{d}P$.

In our case, $J$ admits a partial factorization $J = ZL^iL^e$, with $L^i$ and $L^e$ measurable with respect to $\mathfrak{F}^i$ and $\mathfrak{F}^e$, respectively. Specifically, we can take

$$L^i = \exp\left[\int_{[s,t]} \frac{f(Y_u)}{\sigma \dot{Y}_u} \circ \mathrm{d}W_u + \frac{1}{2}\int_{[s,t]} \left[\frac{f(Y_u)}{\dot{Y}_u} - \left(\frac{f(Y_u)}{\sigma \dot{Y}_u}\right)^2\right] \mathrm{d}u\right\},$$

$$L^e = \exp\left[\int_{]s,t[^c} \frac{f(Y_u)}{\sigma \dot{Y}_u} \circ \mathrm{d}W_u + \frac{1}{2}\int_{]s,t[^c} \left[\frac{f(Y_u)}{\dot{Y}_u} - \left(\frac{f(Y_u)}{\sigma \dot{Y}_u}\right)^2\right] \mathrm{d}u\right\},$$

$$Z = \left|1 - \int_0^1 g(s)\,\mathrm{d}s\right|. \tag{4.30}$$

The quotient in (4.29) is then equal to

$$\frac{E_P[\xi Z L^e/\mathfrak{F}^i]}{E[ZL^e/\mathfrak{F}^i]}.$$

Taking $\xi = \eta(L^e)^{-1}$, with $\eta$ an $\mathfrak{F}^e$-measurable random variable, we obtain that $\{(X_u, \dot{X}_u), u \in [0,1]\}$ is a Markov field iff

$$\Lambda_\eta := \frac{E[\eta Z/\mathfrak{F}^i]}{E[Z/\mathfrak{F}^i]} \tag{4.31}$$

is $\mathfrak{F}^b$-measurable, for every bounded and $\mathfrak{F}^e$-measurable variable $\eta$.

Moreover, interpreting now $\eta$, $Z$, and $\Lambda_\eta$ as random variables on $\Sigma$, and $\mathfrak{F}_i$ as the $\sigma$-field on $\Sigma$ generated by the sets $\{Y: Y_u \in B, B \in \mathfrak{B}(\mathbb{R}), u \in [0,t]\}$ and $\{Y: \dot{Y}_0 \in B, B \in \mathfrak{B}(\mathbb{R})\}$, the condition above can be written

$$\Lambda_\eta := \frac{E_{P_Y}[\eta Z/\mathfrak{F}^i]}{E_{P_Y}[Z/\mathfrak{F}^i]} \text{ is } \mathfrak{F}^b\text{-measurable}, \tag{4.32}$$

where $P_Y$ is the law of $Y: \Omega \to \Sigma$.

Our goal is to prove that, under certain hypotheses on the function $f$ of (1.1), if the process $\{(X_u, X_u), u \in [0,1]\}$ is a Markov field, then $f$ is linear, and that in this case the process is in fact a Markov process. The second claim is easy:

**Theorem 4.4.** *If $f$ is linear, then $\{(X_u, \dot{X}_u), u \in [0,1]\}$ is a Markov process.*

**Proof.** If $f(x) = \alpha x$, the integral equation defining $g$ is

$$g(t) + \int_0^t \frac{\alpha(1 - Y_t) Y_s}{\dot{Y}_t(1 - Y_s)} g(s)\,\mathrm{d}s = 0,$$

and has $g \equiv 0$ as the only solution. Therefore, in this case, $Z \equiv 1$ and $J$ can be written as a product of two random variables measurable with respect to $\sigma\{(Y_u, \dot{Y}_u), u \in [0, t]\}$ and $\sigma\{(Y_u, \dot{Y}_u), u \in [t, 1]\}$, respectively (take $L^{\mathrm{i}}$ and $L^{\mathrm{e}}$ with $s = 0$). Using that $(Y_t, \dot{Y}_t)$ is a Markov process (Proposition 4.1), we obtain $\Lambda_\eta$ is $(Y_t, \dot{Y}_t)$-measurable. Hence, $(Y_t, \dot{Y}_t)$ has the Markov process property also under $Q$, and $(X_t, \dot{X}_t)$ must be a Markov process under $P$.

In fact, if $f$ is linear, we see from expression (4.4) that $D_t u_s$ is a Volterra kernel ($D_t u_s = 0$, for $t > s$). In that case, the Carleman–Fredholm determinant of $I + Du$ is always equal to 1 (see, for instance, Cochran 1972, Section 5.1). This implies directly a factorization of this type for $J$. □

To see the converse, the usual technique, employed in other boundary value settings, is the Malliavin Calculus on $(\Omega, \mathfrak{F}, P)$. Our method will be different. We will perform a calculus of increments directly on the path space of the process $Y$.

Recall that

$$\Sigma = \{y : [0, 1] \to [0, 1] \text{ of class } C^1 \text{ such that } y_0 = 0, y = 1 \text{ and } \dot{y} > 0\}.$$

We consider in $\Sigma$ the topology induced by the norm $\|y\|_\infty + \|\dot{y}\|_\infty$. The law of the random variable $Y : \Omega \to \Sigma$ has the whole set $\Sigma$ as topological support, because $Y$ has a continuous and bijective version. Clearly, the mapping $Z : \Sigma \to \mathbb{R}$ defined by

$$Z(y) = \left| 1 - \int_0^1 g[y](s)\,\mathrm{d}s \right|, \tag{4.33}$$

where $g[y]$ solves (4.9) with $Y = y$, is continuous on $\Sigma$. Moreover, $1 - \int_0^1 g(s)\,\mathrm{d}s$ is always positive or negative, since $I + Du$ is invertible for all $\omega \in \Omega$.

**Proposition 4.5.** *Fix $t \in ]0, 1[$. Let $T_1, T_2 : \Sigma \to \Sigma$ be continuous transformations such that*

$$\overline{T_1(y)}_0 = \overline{T_2(y)}_0 = \dot{y}_0, \qquad \overline{T_1(y)}_t = \overline{T_2(y)}_t = \dot{y}_t \tag{4.34}$$

*and*

$$T_1(y)_s = y_s, \quad \forall s \in [t, 1], \qquad T_2(y)_s = y_s, \quad \forall s \in [0, t]. \tag{4.35}$$

*Assume that $f$ satisfies the hypothesis of Proposition 4.2 and that condition (4.32) holds for the points $s = 0$ and $t$. Then,*

$$Z(y)Z(T_1(T_2(y))) = Z(T_1(y))Z(T_2(y)), \qquad \forall y \in \Sigma. \tag{4.36}$$

**Proof.** Under hypothesis (4.32), we have $\Lambda_\eta(y) = \Lambda_\eta(\dot{y}_0, y_t, \dot{y}_t)$. This implies that, if $T : \Sigma \to \Sigma$ is a transformation verifying (4.34) and (4.35), then $\Lambda_\eta(T(y)) = \Lambda_\eta(y)$. That means,

$$\frac{E[\eta Z/\mathfrak{F}^{\mathrm{i}}](T(y))}{E[Z/\mathfrak{F}^{\mathrm{i}}](T(y))} = \frac{E[\eta Z/\mathfrak{F}^{\mathrm{i}}](y)}{E[Z/\mathfrak{F}^{\mathrm{i}}](y)}, \quad P_Y\text{-a.s.} \tag{4.37}$$

We also have

$$E[Z/\mathfrak{F}^{i}](T_1(y)) = E[Z \circ T_1/\mathfrak{F}^{i}](y), \quad P_Y\text{-a.s.} \tag{4.38}$$

and

$$E[\eta Z/\mathfrak{F}^{i}](T_1(y)) = E[\eta(Z \circ T_1)/\mathfrak{F}^{i}](y), \quad P_Y\text{-a.s.} \tag{4.39}$$

Indeed, we can assume that $Z$ has the form $Z = Z^{i}Z^{e}$, with $Z^{i}$ and $Z^{e}$ $\mathfrak{F}^{i}$ and $\mathfrak{F}^{e}$-measurable, respectively. Then,

$$\begin{aligned} E[\eta Z/\mathfrak{F}^{i}](T_1(y)) &= Z^{i}(T_1(y))E[\eta Z^{e}/\mathfrak{F}^{i}](T_1(y)) \\ &= Z^{i}(T_1(y))E[\eta Z^{e}/\mathfrak{F}^{i}](y) \\ &= E[(Z^{i} \circ T_1)\eta Z^{e}/\mathfrak{F}^{i}](y) \\ &= E[(Z^{i} \circ T_1)\eta(Z^{e} \circ T_1)/\mathfrak{F}^{i}](y) \\ &= E[\eta(Z \circ T_1)/\mathfrak{F}^{i}](y), \end{aligned}$$

where we have used that $E[\eta Z^{e}/\mathfrak{F}^{i}]$ is $\mathfrak{F}^{b}$-measurable (because $\{(Y_u, \dot{Y}_u), u \in [0,1]\}$ is a Markov field) together with conditions (4.34) and (4.35), and that $T_1$ is $\mathfrak{F}^{i}$-measurable (yielding $Z^{e} \circ T_1 = Z^{e}$). This proves (4.39), and the proof of (4.38) is identical. Substituting (4.38) and (4.39) in (4.37) we obtain

$$E[\eta(Z \circ T_1)E[Z/\mathfrak{F}^{i}]/\mathfrak{F}^{i}] = E[\eta Z E[Z \circ T_1/\mathfrak{F}^{i}]/\mathfrak{F}^{i}], \quad P_Y\text{-a.s.}$$

In other words, for all $\xi$ $\mathfrak{F}^{i}$-measurable and for all $\eta$ $\mathfrak{F}^{e}$-measurable,

$$E[\xi\eta(Z \circ T_1)E[Z/\mathfrak{F}^{i}]] = E[\xi\eta Z E[Z \circ T_1/\mathfrak{F}^{i}]], \quad P_Y\text{-a.s.} \tag{4.40}$$

Equality (4.40) remains true substituting $\xi\eta$ by any random variable $\mathfrak{F}^{i} \vee \mathfrak{F}^{e}$-measurable, yielding

$$(Z \circ T_1)E[Z/\mathfrak{F}^{i}] = Z E[Z \circ T_1/\mathfrak{F}^{i}], \quad P_Y\text{-a.s.},$$

and we deduce that $(Z \circ T_1)/Z$ is $\mathfrak{F}^{i}$-measurable. This implies

$$\frac{Z \circ T_1}{Z}(T_2(y)) = \frac{Z \circ T_1}{Z}(y), \quad P_Y\text{-a.s.},$$

and consequently equality (4.36) as it was to be proved. Taking the continuous version of $Z$ defined in (4.33), the equality holds for all $y \in Z$. □

**Proposition 4.6.** *Suppose that $\{(X_t, \dot{X}_t), t \in [0,1]\}$ is a Markov field, and $f$ satisfies the hypothesis of Proposition 4.2. Let $y, y_1, y_2 \in \Sigma$ be such that, for some $t \in ]0,1[$,*

$$\dot{y}(0) = \dot{y}_1(0) = \dot{y}_2(0), \qquad \dot{y}(t) = \dot{y}_1(t) = \dot{y}_2(t),$$

$$y_2(s) = y(s), \quad \forall s \in [0,t], \qquad y_1(s) = y(s), \quad \forall s \in [t,1].$$

*Then,*

$$\left(\int_0^t (g[y](s) - g[y_1](s))\,ds\right)\cdot\left(\int_t^1 (g[y](s) - g[y_2](s))\,ds\right) = 0. \tag{4.41}$$

**Proof.** Denote

$$y_{1,2}(s) := \begin{cases} y_1(s) & \text{if } s \leqslant t, \\ y_2(s) & \text{if } s \geqslant t. \end{cases}$$

Then, (4.32) holds for $s = 0$ and $t \in ]0, 1[$, and (4.36) implies that

$$Z(y_1)Z(y_2) = Z(y)Z(y_{1,2}).$$

As a consequence,

$$\begin{aligned} &\left(1 - \int_0^t g[y_1](s)\,ds - \int_t^1 g[y](s)\,ds\right) \\ &\quad \times \left(1 - \int_0^t g[y](s)\,ds - \int_t^1 g[y_2](s)\,ds\right) \\ &= \left(1 - \int_0^t g[y](s)\,ds - \int_t^1 g[y](s)\,ds\right) \\ &\quad \times \left(1 - \int_0^t g[y_1](s)\,ds - \int_t^1 g[y_2](s)\,ds\right), \end{aligned}$$

which reduces easily to (4.41). □

We finally state a converse of Theorem 4.4.

**Theorem 4.7.** *Let $f : [0, 1] \to \mathbb{R}$ be $C^1$, nonnegative and nondecreasing, with $f(0) = 0$. Let $X_t$ be the solution of (3.1). Then, if the process $\{(X_t, \dot{X}_t), t \in [0, 1]\}$ is a Markov field, $f$ is linear.*

**Proof.** Suppose $f$ is nonlinear. We are going to see that (4.41) leads to a contradiction. For the function $y(t) = t$, we have the associated kernel

$$M(t, s) = \frac{1 - t}{1 - s}[f(t) - f'(t)(t - s)].$$

Set $a := \inf\{x : f(x) - f'(x)x \neq 0\}$. For all $t \in [0, a]$, $M(t, 0) = 0$, and hence $g(t) = 0$. On the other hand, there exists $\delta > 0$ such that $M(t, 0) \neq 0$, for all $t \in ]a, a + \delta]$, implying that $g \neq 0$ on some interval $[c, d] \subset ]a, a + \delta]$.

Fix $\alpha \in [c, d]$ and consider the sequence of functions $y^n : [0, 1] \to [0, 1]$ with derivatives

$$\dot{y}^n(t) = \begin{cases} \dfrac{1}{n} & \text{if } \alpha \leqslant t < \alpha + \dfrac{1}{n}, \\[2mm] 2 - \dfrac{1}{n} & \text{if } \alpha + \dfrac{1}{n} \leqslant t < \alpha + \dfrac{2}{n}, \\[2mm] 1 & \text{if } t < \alpha \text{ or } t \geqslant \alpha + \dfrac{2}{n}. \end{cases}$$

These functions are not in $\Sigma$ but they can be approximated by elements of $\Sigma$. The kernel relative to $y^n$ will be

$$M^n(t,s) = \begin{cases} \dfrac{1-y^n(t)}{1-s}[f(y^n(t)) - f'(y^n(t))(y^n(t)-s)]\dfrac{1}{\dot{y}^n(t)} & \text{if } \alpha < t \leqslant \alpha + \dfrac{2}{n} \text{ and } s<\alpha; \\ \dfrac{1-t}{1-y^n(s)}[f(t) - f'(t)(t-y^n(s))] & \text{if } \alpha + \dfrac{2}{n} < t \text{ and } \alpha < s \leqslant \alpha + \dfrac{2}{n}; \\ M(t,s) & \text{otherwise.} \end{cases}$$

Notice that, as $n \to \infty$,

$$M^n(t,s) \simeq M(t,s)(1 + n\mathbf{1}_{[\alpha,\alpha+1/n[}(t))$$

and therefore the kernels $M^n$ converge to the generalized kernel

$$M(t,s) + M(\alpha,s)\delta_\alpha(t),$$

where $\delta_\alpha$ is Dirac's delta at $\alpha$.

The solution $g^n$ of the integral equation (4.9) with kernel $M^n$ has the representation

$$g^n(t) = \sum_{k=0}^{\infty}(-1)^k \int_{\{t>s_1>\cdots>s_k>0\}} M^n(t,s_1)M^n(s_1,s_2)\cdots M^n(s_k,0)\,\mathrm{d}s_k \ldots \mathrm{d}s_1$$

(the term for $k=0$ should be interpreted as $M^n(t,0)$).

Take $T>\alpha$. Then,

$$\begin{aligned} &\lim_{n\to\infty}\int_0^T g^n(t)\,\mathrm{d}t \\ &= \lim_{n\to\infty}\sum_{k=0}^{\infty}(-1)^k \int_{\{T>t>s_1>\cdots>s_k>0\}} M^n(t,s_1)M^n(s_1,s_2)\cdot\cdots\cdot M^n(s_k,0)\,\mathrm{d}s_k\ldots\mathrm{d}s_1\,\mathrm{d}t \\ &= \sum_{k=0}^{\infty}(-1)^k \int_{\{T>t>s_1>\cdots>s_k>0\}} (M(t,s_1) + M(t,s_1)\delta_\alpha(t))\cdot\cdots\cdot(M(s_k,0) \\ &\quad + M(s_k,0)\delta_\alpha(s_k))\,\mathrm{d}s_k\ldots\mathrm{d}s_1\,\mathrm{d}t. \end{aligned} \tag{4.42}$$

The integrals in (4.42) not involving delta's form a series representation for $\int_0^T g(t)\,\mathrm{d}t$. The other ones are zero unless they involve $\delta_\alpha$ acting upon consecutive variables. The integrals with $\delta_\alpha$ acting upon the variables $s_i$ to $s_j$, for $0 \leqslant i \leqslant j \leqslant k$ (understanding $s_0 = t$), form the series

$$\begin{aligned} &\sum_{k=0}^{\infty}(-1)^k \int_{\{T>t>s_1>\cdots>s_{i-1}>\alpha\}} M(t,s_1)\cdot\cdots\cdot M(s_{i-1},\alpha)\,\mathrm{d}s_{i-1}\ldots\mathrm{d}s_1\,\mathrm{d}t \\ &\quad\times\frac{M(\alpha,\alpha)^{j-i}}{(j-i+1)!}\int_{\{\alpha>s_{j+1}>\cdots>s_k>0\}} M(\alpha,s_{j+1})\cdot\cdots\cdot M(s_k,0)\,\mathrm{d}s_k\ldots\mathrm{d}s_{j+1}, \end{aligned}$$

with the convention that for $i = 0$ the first integral is equal to 1, and for $j = k$ the second one is replaced by $M(\alpha, 0)$.

Therefore, we have

$$\lim_{n\to\infty} \int_0^T g^n(t)\,dt - \int_0^T g(t)\,dt$$

$$= \sum_{k=0}^{\infty} (-1)^k \sum_{i=0}^{k} \sum_{j=i}^{k} \frac{M(\alpha,\alpha)^{j-i}}{(j-i+1)!}$$

$$\times \int_{\{T>t>s_1>\cdots>s_{i-1}>\alpha\}} M(t,s_1)\cdot\cdots\cdot M(s_{i-1},\alpha)\,ds_{i-1}\ldots ds_1\,dt$$

$$\times \int_{\{\alpha>s_{j+1}>\cdots>s_k>0\}} M(\alpha,s_{j+1})\cdot\cdots\cdot M(s_k,0)\,ds_k\ldots ds_{j+1}$$

$$= \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \sum_{k=j}^{\infty} (-1)^k \frac{M(\alpha,\alpha)^{j-i}}{(j-i+1)!}$$

$$\times \int_{\{T>t>s_1>\cdots>s_{i-1}>\alpha\}} M(t,s_1)\cdot\cdots\cdot M(s_{i-1},\alpha)\,ds_{i-1}\ldots ds_1\,dt$$

$$\times \int_{\{\alpha>s_{j+1}>\cdots>s_k>0\}} M(\alpha,s_{j+1})\cdot\cdots\cdot M(s_k,0)\,ds_k\ldots ds_{j+1}$$

$$= g(\alpha) \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} (-1)^j \frac{M(\alpha,\alpha)^{j-i}}{(j-i+1)!}$$

$$\times \int_{\{T>t>s_1>\cdots>s_{i-1}>\alpha\}} M(t,s_1)\cdot\cdots\cdot M(s_{i-1},\alpha)\,ds_{i-1}\ldots ds_1\,dt$$

$$= g(\alpha) \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{l+i} \frac{M(\alpha,\alpha)^{l}}{(l+1)!}$$

$$\times \int_{\{T>t>s_1>\cdots>s_{i-1}>\alpha\}} M(t,s_1)\cdot\cdots\cdot M(s_{i-1},\alpha)\,ds_{i-1}\ldots ds_1\,dt$$

$$= g(\alpha) \sum_{l=0}^{\infty} (-1)^{l+1} \frac{M(\alpha,\alpha)^{l}}{(l+1)!} \sum_{i=0}^{\infty} (-1)^{i+1}$$

$$\times \int_{\{T>t>s_1>\cdots>s_{i-1}>\alpha\}} M(t,s_1)\cdot\cdots\cdot M(s_{i-1},\alpha)\,ds_{i-1}\ldots ds_1\,dt.$$

$$= g(\alpha) \frac{1-e^{-M(\alpha,\alpha)}}{M(\alpha,\alpha)} \sum_{i=0}^{\infty} (-1)^i$$

$$\times \int_{\{T>t>s_1>\cdots>s_{i-1}>\alpha\}} M(t,s_1)\cdot\cdots\cdot M(s_{i-1},\alpha)\,ds_{i-1}\ldots ds_1\,dt.$$

Denote by $\Gamma(\alpha, T)$ the series in this expression. We can perform the same computations between $\beta$ and 1, for $\alpha \leqslant T \leqslant \beta \leqslant 1$. Eq. (4.41) implies then

$$g(\alpha)\frac{1-e^{-f(\alpha)}}{f(\alpha)}\Gamma(\alpha, T)g(\beta)\frac{1-e^{-f(\beta)}}{f(\beta)}\Gamma(\beta, 1) = 0, \tag{4.43}$$

for all $T$ and $\beta$ such that $\alpha \leqslant T \leqslant \beta \leqslant 1$. Since $\lim_{T\searrow\alpha}\Gamma(\alpha, T) = 1$, we obtain that, for $|\beta - \alpha|$ small enough, and taking into account the continuity of $g$, condition (4.43) reduces to $\Gamma(\beta, 1) = 0$.

Denote

$$\Gamma(\beta) := \Gamma(\beta, 1) = 1 + \sum_{i=1}^{\infty}(-1)^i \int_{\{1>s_i>\cdots>s_1>\beta\}} M(s_i, s_{i-1})\cdot\cdots\cdot M(s_1, \beta)\,ds_1 \ldots ds_i. \tag{4.44}$$

Differentiating $\Gamma(\beta)$ yields

$$\begin{aligned}\frac{\partial\Gamma}{\partial\beta} &= -f(\beta)\Bigg[-1+\sum_{i=2}^{\infty}(-1)^i \\ &\quad\times\int_{\{1>s_i>\cdots>s_2>\beta\}} M(s_i, s_{i-1})\cdot\cdots\cdot M(s_2,\beta)\,ds_2\ldots ds_i\Bigg] \\ &\quad+\sum_{i=1}^{\infty}(-1)^i\int_{\{1>s_i>\cdots>s_1>\beta\}} M(s_i, s_{i-1})\cdot\cdots\cdot M(s_2, s_1)\frac{\partial M}{\partial\beta}(s_1,\beta)\,ds_1\ldots ds_i \\ &= f(\beta)\Gamma(\beta) \\ &\quad+\sum_{i=1}^{\infty}(-1)^i\int_{\{1>s_i>\cdots>s_1>\beta\}}\Bigg[\frac{1-s_1}{(1-\beta)^2}(f(s_1)-f'(s_1)(s_1-\beta)) \\ &\quad+\frac{1-s_1}{1-\beta}f'(s_1)\Bigg]M(s_i, s_{i-1})\cdot\cdots\cdot M(s_2, s_1)\,ds_1\ldots ds_i \\ &= f(\beta)\Gamma(\beta) \\ &\quad+\sum_{i=1}^{\infty}(-1)^i\int_{\{1>s_i>\cdots>s_1>\beta\}}\left[\frac{1-s_1}{(1-\beta)^2}f'(s_1)+\frac{1-s_1}{(1-\beta)^2}f(s_1)\right] \\ &\quad\times M(s_i, s_{i-1})\cdot\cdots\cdot M(s_2, s_1)\,ds_1\ldots ds_i\end{aligned}$$

Hence,

$$\begin{aligned}(\Gamma'(\beta) - f(\beta)\Gamma(\beta))(1-\beta)^2 &= \sum_{i=1}^{\infty}(-1)^i \\ &\quad\times\int_{\{1>s_i>\cdots>s_1>\beta\}}\left[\frac{1-s_1}{(1-\beta)^2}f'(s_1)+\frac{1-s_1}{(1-\beta)^2}f(s_1)\right] \\ &\quad\times M(s_i, s_{i-1})\cdot\cdots\cdot M(s_2, s_1)\,ds_1\ldots ds_i\end{aligned}$$

Differentiating again,

$$(\Gamma''(\beta) - f(\beta)\Gamma'(\beta) - f'(\beta)\Gamma(\beta))(1-\beta)^2 - 2(\Gamma'(\beta) - f(\beta)\Gamma(\beta))(1-\beta)$$
$$= -((1-\beta)^2 f'(\beta) + (1-\beta)f(\beta))\Gamma(\beta).$$

Thus, $\Gamma(\beta)$ satisfies the homogeneous linear differential equation

$$(1-\beta)\Gamma''(\beta) + (-f(\beta)(1-\beta) - 2)\Gamma'(\beta) + (f(\beta) - 2f'(\beta)(1-\beta))\Gamma(\beta) = 0$$

for all $\beta \in ]\alpha, 1]$. We have initial conditions $\Gamma(\beta_0) = \Gamma'(\beta_0) = 0$ at some point $\beta_0$ close to $\alpha$. Therefore, $\Gamma \equiv 0$ on $]\alpha, 1[$. But, for (4.44), $\lim_{\beta\to 1}\Gamma(\beta) = 1$, and we get a contradiction. □

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