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Reciprocal Property for a Class of Anticipating Stochastic Differential Equations

A. Alabert* and M.A. Marmolejo†

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193-Bellaterra, Spain

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Abstract. We study a class of one-dimensional stochastic differential equations with boundary conditions by means of a change of variables that reduces the diffusion coefficient to a constant. We obtain a representation of the type $X_t = G(t, Y_t)$, where Y is the solution of the simpler equation. This representation is used to show several properties of the original equation. In particular, our main result is a characterization of the coefficients for which the solution process satisfies a suitable Markov-type property, namely, the reciprocal property.

KEYWORDS: anticipating stochastic differential equations, reciprocal processes

AMS SUBJECT CLASSIFICATION: 60H10, 60J25, 60H07

1. Introduction

It is well known that under appropriate conditions one can reduce a classical Itô stochastic differential equation with a non-degenerate diffusion coefficient to another simpler equation with unit diffusion (see the heuristics for instance in the books by Kloeden and Platen [8, p. 115], or Gihman and Skorohod [6, p. 34]). The transformation is analogous to the "change of unknown function" in o.d.e. practice, and its stochastic counterpart is of course implemented through the Itô change of variables formula.

We want to apply a similar transformation to a class of anticipating stochastic differential equations, by means of an anticipating Itô formula. Specifically,

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we consider the following one-dimensional s.d.e.:

$$(\mathcal{A})\qquad \begin{cases} X_t = X_0 + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\circ dW_s, & t\in[0,1],\\ X_0 = \chi(X_1), \end{cases}$$

where instead of fixing an initial condition X_0, we impose a boundary condition that links the values of the solution at times 0 and 1 through a deterministic function χ. Since X_0 depends strongly on X_1, which in turn depends on the whole evolution of the Wiener process W, the solution will not be adapted to the filtration of W. Therefore, the stochastic integral involved must be understood as an anticipating integral. The circle denotes, as usual, the Stratonovich integral. The corresponding stochastic calculus has undergone an important progress in the last ten years, and provides the necessary tools to treat anticipating problems. In particular, several versions of non-adapted Itô formulae have been established. In Section 2 we summarize the portion of this theory that we need in the present paper.

Equations of type $(\mathcal{A})$ have been considered by several authors: Ocone and Pardoux [13], Donati-Martin [4] and Alabert, Ferrante and Nualart [1], among others. In special cases, existence and uniqueness has been proved inside a certain class of processes, and Markovian type properties of the solution have been studied. The usual Markov property, however, only holds in trivial cases, also because of the strong relation between X_0 and X_1. Instead, the reciprocal property (also called Markov field property), stating the conditional independence of $\{X_t,\ t\in[a,b]\}$ and $\{X_t,\ t\in]a,b[^c\}$ given X_a and X_b seems to be better suited for this kind of problems.

In particular, when $b,\sigma,\chi\colon \mathbf{R}\to\mathbf{R}$ are C^1 functions such that

a) $\sigma>0$, and $x\mapsto \int_0^x \frac{1}{\sigma(\xi)}\,d\xi$ maps $\mathbf{R}$ onto $\mathbf{R}$,

b) $y\mapsto (\frac{b}{\sigma}\circ G^{-1})(y)$ has a bounded derivative,

c) $\chi'\le 0$,

it was proved in [1, Theorems 5.1 and 5.2] that the problem

$$(\mathcal{A}')\qquad \begin{cases} X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\circ dW_s, & t\in[0,1],\\ X_0 = \chi(X_1), \end{cases}$$

admits a unique solution in the Sobolev space of processes $\mathsf{L}^{1,4}_{C,\mathrm{loc}}$, and that this solution is a reciprocal process if and only if $\chi'\equiv 0$ (and in this case it is in fact Markovian) or $b(x) = A\sigma(x) + B\sigma(x)\int_c^x (1/\sigma(y))\,dy$ for some constants $A, B, c \in \mathbf{R}$. This example, and other particular cases studied so far, tell us that equation $(\mathcal{A})$ does not enjoy the reciprocal property in general.

Our main interest in this paper is to characterize the coefficients b and σ for which the reciprocal property is true in the nonautonomous case $(\mathcal{A})$, thus generalizing the results in [1]. We show first that under conditions on $b, \sigma\colon [0,1]\times \mathbf{R} \to \mathbf{R}$ and $\chi\colon \mathbf{R} \to \mathbf{R}$ similar to a), b), c) above (condition c) will be slightly relaxed) the problem $(\mathcal{A})$ has a solution, which can be represented in the form $X_t = G(t, Y_t)$, where $Y = \{Y_t,\ t \in [0,1]\}$ is the solution to

$$(\mathcal{B}) \qquad \begin{cases} Y_t = Y_0 + \int_0^t f(s, Y_s)\,ds + W_t, & t \in [0,1], \\ Y_0 = \psi(Y_1), \end{cases}$$

and the functions G, f and ψ are related with b, σ and χ by

$$\begin{cases} \partial_2 G(t,y) = \sigma(t, G(t,y)), \\ G(t,0) = 0, \end{cases}$$

$$f(t,y) = \frac{b(t, G(t,y)) - \partial_1 G(t,y)}{\sigma(t, G(t,y))} \quad \text{and} \quad \psi(y) = G^{-1}(0, \chi(G(1,y))).$$

In turn, Y can be represented in the form $Y_t(\omega) = \varphi_t(\omega, Y_0(\omega))$, where $Y_0(\omega)$ is a fix point of $y \mapsto \psi(\varphi_1(\omega, y))$, and $\varphi(\omega, y)$ is the unique solution to the integral equation

$$(\mathcal{C}) \quad \varphi_t(\omega, y) = y + \int_0^t f(s, \varphi_s(\omega, y))\,ds + \omega_t, \quad t \in [0,1],$$

for $y \in \mathbf{R}$ and ω a continuous function vanishing at zero. Thus, we can write

$$X_t(\omega) = G(t, \varphi_t(\omega, Y_0(\omega))).$$

We use this representation to characterize the class of boundary value problems $(\mathcal{A})$ for which X is a reciprocal process. Our main result (Theorem 5.2) states that X is reciprocal if and only if either χ is constant or b and σ are related by

$$b(t,x) = \sigma(t,x)\left[\frac{b(t,0)}{\sigma(t,0)} + a(t)\int_0^x \frac{1}{\sigma(t,s)}\,ds + \int_0^x \frac{\partial_1\sigma(t,s)}{\sigma^2(t,s)}\,ds\right],$$

for some function $a(t)$.

We explain now briefly what can be found in each of the sections of this work. In Section 2 we present some results from the anticipating stochastic calculus that we will need in the rest of the paper. In Section 3 we study equation ($\mathcal{B}$), showing that it has a unique solution under Lipschitz and linear growth conditions on f and a one-sided Lipschitz condition on ψ. It is also shown that the solution process belongs to the Sobolev spaces $\mathsf{L}_C^{1,p}$. To this end, we point out previously some properties of equation ($\mathcal{C}$) that are useful in studying ($\mathcal{B}$).

Section 4 is devoted to the study of the general equation ($\mathcal{A}$), assuming the diffusion coefficient is non-degenerate. By means of a change of variable, we obtain a solution to ($\mathcal{A}$) that can be represented as a deterministic function of the solution to ($\mathcal{B}$), and belongs to $\mathsf{L}_{C,\mathrm{loc}}^{1,p}$, for all p. In Section 5 we use this fact to determine in which cases the solution is a reciprocal process. Other properties of equation ($\mathcal{A}$) that can be easily proved using this representation are placed in the final Section 7. We establish the absolute continuity of the law of X_t with respect to Lebesgue measure, a maximal inequality and a comparison result.

A special case of equation ($\mathcal{A}$) appears when the diffusion coefficient is linear in the second variable: $\sigma(t,x) = \sigma(t)x$, with $\sigma(t) > 0$. Strictly speaking, this situation is not covered by the results of Section 4, since $\sigma(t,x)$ vanishes at some points. Nevertheless, under additional conditions, a slight modification of the arguments allows to apply the same change of variable technique to this case as well. This is shown in Section 6. We consider also linear boundary conditions in this section.

We will use the notation $\partial_i f$ for the derivative of a function f with respect to the ith coordinate, and $\partial_{ij} f$ for the second derivative $\partial_i(\partial_j f)$. The symbol ∇F will always mean the Fréchet differential of F, whereas the notation DF is reserved for the weak derivative of F, as defined in Section 2. In several places where we want to prove the weak differentiability of a functional F, we state first the Fréchet differentiability, which is of interest on its own, and then apply Lemma 2.1. Other methods are possible.

2. Preliminaries

Throughout the paper, $W = \{W_t,\ t \in [0,1]\}$ will denote the coordinate process on the classical Wiener space $(\Omega, \mathfrak{F}, \mathsf{P})$. That means, $\Omega = C_0([0,1], \mathbb{R})$ is the Banach space of continuous functions $\omega\colon [0,1] \to \mathbb{R}$ with $\omega(0) = 0$, endowed with the supremum norm, $\mathfrak{F}$ is the Borel σ-field, and P is the standard Wiener measure. In this section we are going to recall some notions of analysis on Wiener space and the Stratonovich anticipating integral. We refer the reader to Norin [10], Nualart [11], or Nualart and Pardoux [12] for details.

Set $H = L^2([0,1], \mathfrak{B}([0,1]), \lambda)$, where $\mathfrak{B}([0,1])$ and λ are the Borel σ-field and the Lebesgue measure on $[0,1]$, respectively, and let H_1 denote the Hilbert space of all functions on $[0,1]$ with square integrable derivative, which is densely embedded in Ω. The isomorphism between H and H_1 is given by $h \mapsto \int_0^{\cdot} h$.

A *smooth functional* is a mapping $F: \Omega \to \mathbb{R}$ of the form

$$F = f(W_{t_1}, W_{t_2}, \ldots, W_{t_n}),$$

where $n \geq 1$ and $f \in C^\infty$ has polynomial growth, together with all its derivatives. Denote by $\mathcal{S}$ the set of smooth functionals.

If $F \in \mathcal{S}$, the *weak derivative* of F is the stochastic process $\{D_t F,\ t \in [0,1]\}$ defined by

$$D_t F(\omega) = \sum_{i=1}^{n} \partial_i f(W_{t_1}(\omega), W_{t_2}(\omega), \ldots, W_{t_n}(\omega)) \mathbf{1}_{[0,t_i]}(t).$$

As a random variable, DF belongs to $\bigcap_p L^p(\Omega; H)$, that means, $\mathsf{E}[\|DF\|_H^p] < \infty$, for all $p \geq 1$. The operators

$$D: \mathcal{S} \subseteq L^p(\Omega) \to L^p(\Omega; H), \tag{2.1}$$

$p \geq 1$, are closable, and we denote by $\mathsf{D}^{1,p}$ the closure of $\mathcal{S}$ in $L^p(\Omega)$ with respect to the graph norm

$$\|F\|_{1,p} = [\mathsf{E}(|F|^p) + \mathsf{E}(\|DF\|_H^p)]^{1/p}.$$

Iterating D, one obtains the whole class of Sobolev spaces $\mathsf{D}^{k,p}$.

It is easy to see from the definition that the weak derivative DF of a smooth functional is related to the Fréchet derivative ∇F in the following way:

$$\langle DF(\omega), \dot{h} \rangle_H = {}_{\Omega^*}\langle \nabla F(\omega), h \rangle_\Omega, \quad \forall h \in H_1,$$

where ${}_{\Omega^*}\langle \cdot, \cdot \rangle_\Omega$ denotes the dual pairing. In other words,

$$D_t F(\omega) = [\nabla F(\omega)](]t, 1]),$$

if we consider $\nabla F(\omega)$ as a bounded Borel measure on $[0,1]$. The following lemma follows from the results of Sugita [15].

Lemma 2.1. *Let $F: \Omega \to \mathbb{R}$ be a Lipschitz and Fréchet differentiable mapping. If $F \in L^2(\Omega)$ and $[\nabla F(\omega)](]t, 1]) \in L^2(\Omega; H)$, then $F \in \mathsf{D}^{1,2}$ and*

$$D_t F(\omega) = [\nabla F(\omega)](]t, 1]).$$

Let $\mathsf{D}^{k,p}_{\text{loc}}$ denote the set of random variables F such that there is a sequence $\{(\Omega_n, F_n)\}_{n \in \mathbb{N}}$ in $\mathfrak{F} \times \mathsf{D}^{k,p}$ satisfying $\Omega_n \nearrow \Omega$, a.s. and $F = F_n$ a.s. on Ω_n. The operators D are local, and this allows to define DF for all $F \in \mathsf{D}^{k,p}_{\text{loc}}$ without ambiguity, setting $DF = DF_n$ on Ω_n.

The following properties of the derivative operator will be used later (see Propositions 1.2.2, 1.5.5 and Theorem 2.1.3 in [11] for the proofs).

Lemma 2.2. (1) *Let $\varphi: \mathbf{R}^m \to \mathbf{R}$ be of class C^1 with bounded partial derivatives, and fix $p \geq 1$. Suppose $F = (F^1, F^2, \ldots, F^m)$ is a random vector whose components belong to $\mathsf{D}^{1,p}$. Then $\varphi(F) \in \mathsf{D}^{1,p}$, and*

$$D(\varphi(F)) = \sum_{i=1}^{m} \partial_i \varphi(F) DF^i.$$

(2) *Let F be a random variable in $\mathsf{D}^{1,\alpha}$, with $\alpha > 1$. Suppose $F \in L^p(\Omega)$ and $DF \in L^p(\Omega; H)$ for some $p > \alpha$. Then $F \in \mathsf{D}^{1,p}$.*

(3) *If $F \in \mathsf{D}^{1,1}_{\mathrm{loc}}$ and $\|DF\|_H > 0$ a.s., then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$.*

We shall introduce also some spaces of random processes: $\mathsf{L}^{1,p} := L^p([0,1]; \mathsf{D}^{1,p})$; $\mathsf{L}^{1,p}_C$ is the class of processes $u \in \mathsf{L}^{1,p}$ such that there is a version of the derivative Du with the properties:

a) the sets of $L^p(\Omega)$-valued functions $\{s \mapsto D_{t\vee s} u_{t\wedge s}\}_{t\in[0,1]}$ and $\{s \mapsto D_{t\wedge s} u_{t\vee s}\}_{t\in[0,1]}$ are equicontinuous,

b) $\operatorname*{ess\,sup}_{s,t\in[0,1]} \mathsf{E}(|D_s u_t|^p) < \infty$;

$\mathsf{L}^{1,p}_{\mathrm{loc}}$ is the set of measurable processes u for which there exists a sequence $\{(\Omega_n, u_n)\}_{n\in\mathbb{N}}$ in $\mathfrak{F} \times \mathsf{L}^{1,p}$ such that $\Omega_n \nearrow \Omega$, a.s. and $u = u_n$ a.s. on $[0,1] \times \Omega_n$; finally, the space $\mathsf{L}^{1,p}_{C,\mathrm{loc}}$ is defined analogously, imposing $u_n \in \mathsf{L}^{1,p}_C$.

If $u \in \mathsf{L}^{1,2}_C$, the following limits exist in $L^2(\Omega)$, uniformly in t:

$$D^+_t u_t = \lim_{\epsilon\downarrow 0} D_t u_{t+\epsilon}, \qquad D^-_t u_t = \lim_{\epsilon\downarrow 0} D_t u_{t-\epsilon}.$$

We recall now the usual definition of the anticipating Stratonovich integral. Given a measurable process $u = \{u_t, t \in [0,1]\}$ with $\int_0^1 |u_t|\, dt < \infty$ a.s., define

$$S^\pi_t = \sum_{i=0}^{n-1} \Big[\frac{1}{t_{i+1} - t_i}\Big] \Big(\int_{t_i\wedge t}^{t_{i+1}\wedge t} u_s\, ds \Big) (W(t_{i+1}) - W(t_i)), \quad t \in [0,1],$$

where π is a partition $\{0 = t_0 < t_1 < \ldots < t_n = 1\}$ of $[0,1]$. We say that u is Stratonovich integrable on $[0,1]$ if for any $t \in [0,1]$, S^π_t converge in probability to a limit S_t when $|\pi| \searrow 0$. In that case, S_t is called the Stratonovich integral of u on $[0,t]$, and will be denoted by $\int_0^t u_s \circ dW_s$. It can be seen that if $u \in \mathsf{L}^{1,2}_{C,\mathrm{loc}}$, or if u is continuous and with bounded variation a.s., then u is Stratonovich integrable.

We will make a fundamental use of the following Itô-type formula for the Stratonovich integral, which is a particular case of Theorem 7.10 of Nualart and Pardoux [12].

Theorem 2.1. *Let $\Phi: \mathbb{R}^3 \to \mathbb{R}$ be a continuous function such that the derivatives $\partial_1\Phi$, $\partial_2\Phi$, $\partial_3\Phi$, $\partial_{12}\Phi$, $\partial_{23}\Phi$ and $\partial_{33}\Phi$ are also continuous functions. Let $u \in \mathbb{L}^{2,2} \cap \mathbb{L}^{1,2}_C$ satisfy the following conditions.*

a) *There exists $p > 4$ such that*

$$\int_0^1\int_0^1 |\mathsf{E}(D_s u_t)|^p \, ds\, dt + \mathsf{E}\int_0^1\int_0^1\int_0^1 |D_r D_s u_t|^p \, dr\, ds\, dt < \infty.$$

b)

$$\{D_t^+ u_t + D_t^- u_t,\ t \in [0,1]\} \in \mathbb{L}^{1,2}$$

and

$$\sup_{t\in[0,1]} \mathsf{E}\int_0^1 |D_t(D_s^+ u_s + D_s^- u_s)|^4 \, ds < \infty.$$

Let $\{V_t^i,\ t \in [0,1]\}$, $i = 1, 2$, be continuous processes in $\mathbb{L}^{1,2}$ with finite variation a.s., such that

c) $\mathsf{E}\int_0^1\int_0^1 (D_s V_t^i)^4 \, ds\, dt < \infty, \quad i = 1, 2,$

d) *the mapping $t \to D_s V_t^i$, $i = 1, 2$, from $[0,1]$ into $L^4(\Omega)$ is continuous, uniformly with respect to s.*

Set $U_t := \int_0^t u_s \circ dW_s$.
Then

$$\Phi(V_t^1, V_t^2; U_t) = \Phi(V_0^1, V_0^2; 0) + \int_0^t \partial_1\Phi(V_s^1, V_s^2; U_s)\, dV_s^1 + \int_0^t \partial_2\Phi(V_s^1, V_s^2; U_s)\, dV_s^2 + \int_0^t \partial_3\Phi(V_s^1, V_s^2; U_s) \circ dW_s. \tag{2.2}$$

3. Constant diffusion coefficient

In this section we study some aspects of the s.d.e. with boundary conditions

$$(\mathcal{B}) \quad \begin{cases} Y_t = Y_0 + \int_0^t f(s, Y_s)\, ds + W_t, & t \in [0,1], \\ Y_0 = \psi(Y_1). \end{cases}$$

In [1] it was proved that under Lipschitz and linear growth conditions on f (assumptions (H_1) and (H_2) below), and assuming ψ is a nonincreasing function, there exists a unique solution to $(\mathcal{B})$, which solves the equation pathwise. When f and ψ are of class C^1, it is pointed out without proof that the solution belongs to the space $\mathsf{L}^{1,4}_{C,\mathrm{loc}}$.

Here we are going to precise those arguments with a slightly weaker condition on ψ, and to obtain that in the C^1 case the solution belongs in fact to the smaller space $\bigcap_p \mathsf{L}^{1,p}_C$.

Consider first the deterministic integral equation with initial condition

$$(\mathcal{C}) \quad \varphi_t(\omega, y) = y + \int_0^t f(s, \varphi_s(\omega, y))\, ds + \omega_t, \quad t \in [0,1],$$

where $\omega \in \Omega$ and $y \in \mathbf{R}$.

Proposition 3.1. *Let $f\colon [0,1] \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions.*

(H_1) *There exists a constant $K > 0$ such that for every $t \in [0,1]$ and every $x, y \in \mathbf{R}$, $|f(t,x) - f(t,y)| \le K|x-y|$.*

(H_2) $\sup_{t\in[0,1]} |f(t,0)| = M < \infty$.

Then, for every $\omega \in \Omega$ and every $y \in \mathbf{R}$, there exists a unique continuous function $\varphi(\omega, y)$ satisfying equation $(\mathcal{C})$. Moreover, for each $\omega, \omega^1, \omega^2 \in \Omega$ and each $y, y_1, y_2 \in \mathbf{R}$,

(1) $\sup_{t\in[0,1]} |\varphi_t(\omega, y)| \le [|y| + \|\omega\|_\infty + M]e^K$.

(2) $\sup_{t\in[0,1]} |\varphi_t(\omega^1, y_1) - \varphi_t(\omega^2, y_2)| \le [|y_1 - y_2| + \|\omega^1 - \omega^2\|_\infty]e^K$.

Proof. The main statement can be proved by the usual Picard iterations as in the classical case when ω is differentiable (see [1] for a more general situation, where ds is replaced by $\mu(ds)$, with μ a finite measure on $[0,1]$).

(1) and (2) follow from Gronwall inequality. □

We have also the following additional properties, whose proof is straightforward, when we let y and ω to vary.

Proposition 3.2. *Under assumptions* (H_1) *and* (H_2) *of Proposition* 3.1, *we have moreover:*

(3) *For every $t \in [0,1]$ and every $\omega \in \Omega$, the function $y \mapsto \varphi_t(\omega, y)$ is strictly increasing.*

(4) *For every $y \in \mathbf{R}$, the mapping $\omega \mapsto \varphi(\omega, y)$ is a homeomorphism between Ω and the space $C_0^y = \{x \in C([0,1], \mathbf{R}) : x_0 = y\}$.*

(5) *The mapping $(\omega, y) \mapsto \varphi(\omega, y)$ from $\Omega \times \mathbf{R}$ into $C([0,1], \mathbf{R})$ is a homeomorphism.*

Under differentiability assumptions on f, we obtain differentiability properties of the solution to $(\mathcal{C})$.

Proposition 3.3. *Assume $f: [0,1] \times \mathbf{R} \to \mathbf{R}$ satisfies the following conditions.*

($\mathbf{H}_1'$) *f and $\partial_2 f$ are continuous functions.*

($\mathbf{H}_2'$) *There exists a constant $K > 0$ such that $\sup_{t \in [0,1], y \in \mathbf{R}} |\partial_2 f(t, y)| \leq K$.*

Then, with the notations of Proposition 3.1, we have: for every $t \in [0,1]$, the mapping $(\omega, y) \mapsto \varphi_t(\omega, y)$ is continuously Fréchet differentiable, its Fréchet differential $\nabla\varphi_t(\omega, y)$, as an element of the space $\mathcal{L}(\Omega \times \mathbf{R}, \mathbf{R})$ of continuous linear mappings from $\Omega \times \mathbf{R}$ to $\mathbf{R}$, satisfies

$$[\nabla\varphi_t(\omega, y)](h, \xi) = \xi + \int_0^t \partial_2 f(s, \varphi_s(\omega, y))[\nabla\varphi_s(\omega, y)](h, \xi)\, ds + h_t, \tag{3.1}$$

for all $(h, \xi) \in \Omega \times \mathbf{R}$, and $|[\nabla\varphi_t(\omega, y)](h, \xi)| \leq \exp K(\|h\|_\infty + |\xi|)$.

In particular, if φ_t' denotes the derivative of φ_t with respect to the real variable y, we obtain

$$\varphi_t'(\omega, y) = \exp\Big\{ \int_0^t \partial_2 f(r, \varphi_r(\omega, y))\, dr \Big\}. \tag{3.2}$$

Proof. Fix $(\omega, y) \in \Omega \times \mathbf{R}$. ($\mathrm{H}_1'$) and ($\mathrm{H}_2'$) imply that for each $(h, \xi) \in \Omega \times \mathbf{R}$ there exists a unique continuous function $\mathcal{L}_t(h, \xi)$ solving

$$\mathcal{L}_t(h, \xi) = \xi + \int_0^t \partial_2 f(s, \varphi_s(\omega, y))\mathcal{L}_s(h, \xi)\, ds + h_t, \quad t \in [0,1].$$

By Gronwall inequality,

$$|\mathcal{L}_t(h, \xi)| \leq [|\xi| + \|h\|_\infty]e^K, \quad t \in [0,1],$$

so that $(h, \xi) \mapsto \mathcal{L}_t(h, \xi)$ is a linear continuous functional.

Let us check that, for every $t \in [0,1]$,

$$\varepsilon_t(\omega, h; y, \xi) := \varphi_t(\omega + h, y + \xi) - \varphi_t(\omega, y) - \mathcal{L}_t(h, \xi) = o(\|(h, \xi)\|)$$

when $\|(h,\xi)\| \to 0$. Indeed, we have

$$\begin{aligned}
\varepsilon_t(\omega,h;y,\xi) &= \int_0^t [f(s,\varphi_s(\omega+h,y+\xi)) - f(s,\varphi_s(\omega,y))]\,ds \\
&\quad - \int_0^t \partial_2 f(s,\varphi_s(\omega,y))\mathcal{L}_s(h,\xi)\,ds \\
&= \int_0^t \big[\partial_2 f(s,i_s) - \partial_2 f(s,\varphi_s(\omega,y))\big]\big[\varphi_s(\omega+h,y+\xi) \\
&\quad -\varphi_s(\omega,y)\big]\,ds \\
&\quad + \int_0^t \partial_2 f(s,\varphi_s(\omega,y))\varepsilon_s(\omega,h;y,\xi)\,ds,
\end{aligned}$$

where i_s is some point between $\varphi_s(\omega+h,y+\xi)$ and $\varphi_s(\omega,y)$. Fix $\varepsilon>0$, and denote $\alpha=\inf_{0\le s\le 1}\varphi_s(\omega,y)$, $\beta=\sup_{0\le s\le 1}\varphi_s(\omega,y)$, and $C=[0,1]\times[\alpha-1,\beta+1]$. There exists $0<\delta<1$ such that $(t,x),(t',x')\in C$ and $\max\{|t-t'|,|x-x'|\}<\delta$ imply

$$|\partial_2 f(t,x)-\partial_2 f(t',x')|<\varepsilon.$$

By Proposition 3.1 (2), if $\exp K\|(h,\xi)\|<\delta$,

$$|\,i_s-\varphi_s(\omega,y)|\le|\,\varphi_s(\omega+h,y+\xi)-\varphi_s(\omega,\xi)|\le e^K\|(h,\xi)\|<1,$$

from which $\alpha-1\le i_s\le\beta+1$, for all $s\in[0,1]$, and we deduce that

$$|\,\varepsilon_t(\omega,h;y,\xi)|\le\varepsilon e^K\|(h,\xi)\|+K\int_0^t|\,\varepsilon_s(\omega,h;y,\xi)|\,ds,\quad t\in[0,1].$$

By Gronwall inequality again, $|\varepsilon_t(\omega,h;y,\xi)|\le\varepsilon\exp(2K)\|(h,\xi)\|$. This shows that for every $t\in[0,1]$, the mapping $(\omega,y)\mapsto\varphi_t(\omega,y)$ is Fréchet differentiable and its Fréchet differential $\nabla\varphi_t(\omega,y)$ satisfies (3.1). With similar arguments, one can show the continuity of $(\omega,y)\mapsto\nabla\varphi_t(\omega,y)$. □

In the next proposition we compute the weak derivative of the random variables $\varphi_t(\cdot,y)$ and show that the processes $\varphi_\cdot(\cdot,y)$ belong to the spaces $\mathsf{L}_C^{1,p}$. The fact that $\varphi_t(\cdot,y)\in\mathsf{D}^{1,p}$ can also be proved using Picard iterations, following Nualart [11], Theorem 2.2.1. Here we will use the Fréchet differentiability of $\varphi_t(\cdot,y)$ and Lemmas 2.1 and 2.2.

Proposition 3.4. *Assume* $f\colon [0,1]\times \mathbf{R} \to \mathbf{R}$ *verifies hypotheses* (H_1') *and* (H_2'). *Then, for every* $y \in \mathbf{R}$, $\varphi_{\cdot}(\cdot, y) \in \bigcap_p \mathsf{L}_C^{1,p}$, *and*

$$D_s\varphi_t(\omega, y) = \mathbf{1}_{[0,t]}(s) \exp\Big\{ \int_s^t \partial_2 f(r, \varphi_r(\omega, y))\, dr \Big\}.$$

Proof. We have $\varphi_t(\cdot, y) \in \bigcap_p L^p(\Omega)$. This is a consequence of (1) of Proposition 3.1 and Doob maximal inequality applied to the supremum of the Wiener process. Moreover, by Proposition 3.1 (2), the mapping $\omega \mapsto \varphi_t(\omega, y)$ is Lipschitz.

On the other hand, we know from Proposition 3.3 that $\varphi_t(\cdot, y)$ is Fréchet differentiable and, denoting by ∇_1 the Fréchet differentiation with respect to ω, we have

$$[\nabla_1\varphi_t(\omega, y)](h) = \int_0^t \partial_2 f(s, \varphi_s(\omega, y))[\nabla_1\varphi_s(\omega, y)](h)\, ds + \int_0^t \dot{h}(s)\, ds,$$

for every $h(s) = \int_0^s \dot{h}(r)\, dr \in H_1$. That means,

$$[\nabla_1\varphi_t(\omega, y)](h) = \int_0^1 \dot{h}(s)\mathbf{1}_{[0,t]}(s) \exp\Big\{ \int_s^t \partial_2 f(r, \varphi_r(\omega, y))\, dr \Big\}\, ds,$$

and therefore, as a measure (see Section 2),

$$[\nabla_1\varphi_t(\omega, y)](]s, 1]) = \mathbf{1}_{[0,t]}(s) \exp\Big\{ \int_s^t \partial_2 f(r, \varphi_r(\omega, y))\, dr \Big\}$$

for almost all $r \in [0, 1]$. This function is bounded by $\exp(K)$; hence, according to Lemma 2.1, $\varphi_t(\cdot, y) \in \mathsf{D}^{1,2}$ and

$$D_s\varphi_t(\omega, y) = \mathbf{1}_{[0,t]}(s) \exp\Big\{ \int_s^t \partial_2 f(r, \varphi_r(\omega, y))\, dr \Big\}. \tag{3.3}$$

Using (2) of Lemma 2.2, we conclude $\varphi_t(\cdot, y) \in \bigcap_p \mathsf{D}^{1,p}$.

Let us check that $\varphi_{\cdot}(\cdot, y) \in \mathsf{L}_C^{1,p}$. From (3.3), it is obvious that this process belongs to $\bigcap_p \mathsf{L}^{1,p}$. On the other hand, the sets

$$\{s \mapsto D_{t\vee s}\varphi_{t\wedge s}(\cdot, y)\}_{t\in[0,1]} \quad \text{and} \quad \{s \mapsto D_{t\wedge s}\varphi_{t\vee s}(\cdot, y)\}_{t\in[0,1]}$$

are equicontinuous, since

$$D_{t\vee s}\varphi_{t\wedge s}(\cdot, y) \equiv 0$$

and

$$|D_{t\wedge s_2}\varphi_{t\vee s_2}(\cdot,y) - D_{t\wedge s_1}\varphi_{t\vee s_1}(\cdot,y)| \le Ke^K|s_2 - s_1|.$$

Moreover,

$$\operatorname*{ess\,sup}_{s,t\in[0,1]} \mathsf{E}[|D_s\varphi_t(\cdot,y)|^p] \le e^{pK} < \infty.$$

□

We turn now to the boundary value problem $(\mathcal{B})$.

Theorem 3.1. *Assume $f\colon[0,1]\times\mathbf{R}\to\mathbf{R}$ satisfies conditions* (H$_1$) *and* (H$_2$) *of Proposition* 3.1 *with Lipschitz constant K, and that*

(H$_3$) *$\psi\colon\mathbf{R}\to\mathbf{R}$ is continuous and enjoys the following one-sided Lipschitz condition:*

$$x > y \Rightarrow \psi(x) - \psi(y) \le \eta\cdot(x-y),$$

with $\eta < \exp(-K)$.

Then, for every $\omega\in\Omega$, the problem $(\mathcal{B})$ has a unique solution.

Proof. From (H$_3$), Proposition 3.1 (2) and Proposition 3.2 (3), it is immediate that, if $x > y$,

$$\frac{[\psi\circ\varphi_1](\omega,x) - [\psi\circ\varphi_1](\omega,y)}{x-y} < 1-\varepsilon,$$

for some $\varepsilon > 0$, and this implies that $x\mapsto\psi(\varphi_1(\omega,x))$ intersects the graph of the identity exactly once. Therefore, if Y_0 is the unique fixed point of this mapping, $Y_t(\omega) = \varphi_t(\omega, Y_0(\omega))$ solves the equation. Moreover, if Z is another solution, we must have

$$Z_t(\omega) = \varphi_t(\omega, Z_0(\omega)) = \varphi_t(\omega, Y_0(\omega)) = Y_t(\omega),$$

for each ω and t. □

From Propositions 3.3 and 3.4, we can derive corresponding properties for equation $(\mathcal{B})$. This is the content of the next two propositions.

Proposition 3.5. *Let $f\colon[0,1]\times\mathbf{R}\to\mathbf{R}$ and $\psi\colon\mathbf{R}\to\mathbf{R}$ satisfy* (H$'_1$), (H$'_2$) *and* (H$'_3$), *where:*

(H$'_3$) *ψ is a C^1 function and $\psi' \le \eta < \exp(-K)$.*

Let $Y = \{Y_t,\ t\in[0,1]\}$ be the unique solution to $(\mathcal{B})$. Then, for every $t\in[0,1]$, Y_t is Fréchet differentiable and its Fréchet derivative $[\nabla Y_t](\omega)$ at ω satisfies:

$$\nabla Y_t(\omega)[h] = \nabla Y_0(\omega)[h] + \int_0^t \partial_2 f(s, Y_s(\omega))\nabla Y_s(\omega)[h]\,ds + h_t, \quad h\in\Omega. \quad (3.4)$$

Proof. $Y_0(\omega)$ is the unique solution y to $S(\omega, y) := y - \psi(\varphi_1(\omega, y)) = 0$. The derivative of S with respect to y is strictly positive, because of (H_3') and Proposition (3.3). Thus, from the Implicit Function Theorem, Y_0 is differentiable and

$$\nabla Y_0(\omega)[h] = \Big[\frac{\psi'(\varphi_1(\omega, Y_0(\omega)))}{1 - \psi'(\varphi_1(\omega, Y_0(\omega)))\varphi_1'(\omega, Y_0(\omega))}\Big] \nabla_1 \varphi_1(\omega, Y_0(\omega))[h],$$
$$h \in \Omega. \tag{3.5}$$

On the other hand, we know that the mapping $(\omega, y) \mapsto \varphi_t(\omega, y)$ is Fréchet differentiable and (3.1) holds. Thus, by the chain rule, we obtain the differentiability of $Y_t(\omega) = \varphi_t(\omega, Y_0(\omega))$ and the variational equation (3.4). □

Proposition 3.6. *Let $f: [0, 1] \times \mathbf{R} \to \mathbf{R}$ and $\psi: \mathbf{R} \to \mathbf{R}$ satisfy conditions* (H_1'), (H_2'), (H_3'). *Then $Y \in \bigcap_p \mathbb{L}_C^{1,p}$ and*

$$D_s Y_t = D_s Y_0 \exp\Big\{\int_0^t \partial_2 f(r, Y_r)\, dr\Big\} + \mathbf{1}_{[0,t]}(s) \exp\Big\{\int_s^t \partial_2 f(r, Y_r)\, dr\Big\}, \tag{3.6}$$

where

$$D_s Y_0(\omega) = \frac{\psi'(Y_1(\omega))}{1 - \psi'(Y_1(\omega))\varphi_1'(\omega, Y_0(\omega))} \exp\Big\{\int_s^1 \partial_2 f(r, Y_r(\omega))\, dr\Big\}. \tag{3.7}$$

Proof. We are going to apply again Lemma 2.1, first to Y_0 and then to $Y_t(\omega) = \varphi_t(\omega, Y_0(\omega))$, $t > 0$. From (3.5), we have

$$\|\nabla Y_0(\omega)\| = \Big|\frac{\psi'(\varphi_1(\omega, Y_0(\omega)))}{1 - \psi'(\varphi_1(\omega, Y_0(\omega)))\varphi_1'(\omega, Y_0(\omega))}\Big| \|\nabla_1 \varphi_1(\omega, Y_0(\omega))\|. \tag{3.8}$$

From Proposition 3.3, we find that

$$\|\nabla_1 \varphi_1(\omega, y)\| \le e^K \quad \text{and} \quad \varphi_1'(\omega, y) \ge e^{-K}.$$

If $\psi'(\varphi_1(\omega, Y_0(\omega))) \le 0$, the absolute value in (3.8) can be bounded by $\exp(K)$. If, on the contrary, $\psi'(\varphi_1(\omega, Y_0(\omega)) > 0$, then it can be bounded by $(\exp(-K))/(1 - \eta \exp(K))$. In any case, we obtain $\|\nabla Y_0(\omega)\| \le C$, for some constant C. By the Mean Value Theorem, Y_0 is Lipschitz and $|Y_0(\omega)| \le C \cdot \|\omega\|_\infty + |Y_0(0)|$. We conclude that $Y_0 \in L^p(\Omega)$, for all p.

From (3.5) again and the computations in the proof of Proposition 3.4, we have

$$[\nabla Y_0(\omega)](h) = \frac{\psi'(Y_1(\omega))}{1 - \psi'(Y_1(\omega))\varphi_1'(\omega, Y_0(\omega))} \int_0^1 \dot{h}(s) \exp\Big\{\int_s^1 \partial_2 f(r, Y_r(\omega))\, dr\Big\}\, ds.$$

Proceeding as in that proof, we obtain that the function $s \mapsto [\nabla Y_0(\omega)](]s,1])$ is bounded uniformly in ω. From Lemma 2.1, $Y_0 \in \mathsf{D}^{1,2}$. Applying (2) of Lemma 2.2, we conclude $Y_0 \in \bigcap_p \mathsf{D}^{1,p}$ and that (3.7) holds.

We can perform a similar computation for Y_t:

$$|Y_t(\omega)-Y_t(\tilde{\omega})| = |\varphi_t(\omega, Y_0(\omega))-\varphi_t(\tilde{\omega}, Y_0(\tilde{\omega}))| \leq \big[|Y_0(\omega)-Y_0(\tilde{\omega})|+\|\omega-\tilde{\omega}\|_\infty\big]e^K,$$

that means, Y_t is Lipschitz in ω. From (1) of Proposition 3.1, $Y_t \in L^p(\Omega)$. Finally one obtains from Proposition 3.5 after easy computations that

$$\begin{aligned}[\nabla Y_t(\omega)](]s,1]) &= D_s Y_0(\omega) \exp\Big\{ \int_0^t \partial_2 f(r, Y_r(\omega))\, dr \Big\} \\ &\quad + \mathbf{1}_{[0,t]}(s) \exp\Big\{ \int_s^t \partial_2 f(r, Y_r(\omega))\, dr \Big\},\end{aligned}$$

and from here we have that this random variable is in $L^p(\Omega; H)$ for all p. We again conclude from Lemma 2.1 and Lemma 2.2 that $Y_t \in \bigcap_p \mathsf{D}^{1,p}$.

To show that $Y \in \mathsf{L}_C^{1,p}$, one can prove easily that

$$\begin{aligned}|D_{s_2\vee t} Y_{s_2\wedge t}(\omega) - D_{s_1\vee t} Y_{s_1\wedge t}(\omega)| &\leq Ke^K|s_2 - s_1|, \\ |D_{s_2\wedge t} Y_{s_2\vee t}(\omega) - D_{s_1\wedge t} Y_{s_1\vee t}(\omega)| &\leq 2Ke^K|s_2 - s_1|, \\ \operatorname*{ess\,sup}_{s,t\in[0,1]} \mathsf{E}(|D_s Y_t|^p) &< 2^p e^{2Kp} < \infty. \end{aligned} \tag{3.9}$$

□

4. General (non-degenerate) diffusion coefficient

In this section we study the boundary value problem

$$(\mathcal{A}) \quad \begin{cases} X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s) \circ dW_s, & t \in [0,1], \\ X_0 = \chi(X_1). \end{cases}$$

By a solution to $(\mathcal{A})$ we mean a continuous stochastic process $X = \{X_t,\ t \in [0,1]\}$ such that $\sigma(\cdot, X_\cdot)$ is Stratonovich integrable on $[0,1]$ and the equalities in $(\mathcal{A})$ hold true almost surely. Under appropriate conditions on b, σ and χ, we will prove the existence of a solution belonging to $\bigcap_p \mathsf{L}_{C,\mathrm{loc}}^{1,p}$. We will assume the following conditions on $\sigma\colon [0,1] \times \mathbb{R} \to \mathbb{R}$:

$(\mathbf{H}_4)$ $\sigma(t,x) > 0$ for every $x \in \mathbb{R}$ and every $t \in [0,1]$.

(H_5) For every $t \in [0,1]$,

$$\int_0^\infty \frac{1}{\sigma(t,x)}\,dx = \int_{-\infty}^0 \frac{1}{\sigma(t,x)}\,dx = \infty.$$

(H_6) σ, $\partial_1\sigma$ and $\partial_2\sigma$ are continuous functions.

Let $G: [0,1] \times \mathbf{R} \to \mathbf{R}$ be the function defined by the differential equation

$$\begin{cases} \partial_2 G(t,y) = \sigma(t, G(t,y)), \\ G(t,0) = 0. \end{cases}$$

Assumptions (H_4)–(H_6) imply that:

(1) for any $t \in [0,1]$, the function $G(t,\cdot): \mathbf{R} \to \mathbf{R}$ is well defined and it is the inverse of

$$G^{-1}(t,x) = \int_0^x \frac{1}{\sigma(t,\xi)}\,d\xi.$$

(2) $\partial_1 G(t,y)$ is continuous and differentiable with respect to y. Moreover, $\partial_{21}G(t,y) = \partial_{12}G(t,y)$ (see Theorem 6.3.1 in [3]). We obtain the differential equation

$$\partial_2[\partial_1 G(t,y)] = \partial_2\sigma(t,G(t,y))\partial_1 G(t,y) + \partial_1\sigma(t,G(t,y)).$$

Taking into account that $\partial_1 G(t,0) \equiv 0$, we have

$$\partial_1 G(t,y) = \int_0^y \partial_1\sigma(t,G(t,s)) \exp\left\{ \int_s^y \partial_2\sigma(t,G(t,\xi))\,d\xi \right\} ds.$$

(3) $\partial_{22}G(t,y) = \partial_2\sigma(t,G(t,y))\sigma(t,G(t,y))$.

With the preceding notations, let now $b: [0,1] \times \mathbf{R} \to \mathbf{R}$ be such that:

(H_7) The function $f: [0,1] \times \mathbf{R} \to \mathbf{R}$ defined by

$$f(t,y) = \frac{b(t,G(t,y)) - \partial_1 G(t,y)}{\sigma(t,G(t,y))}, \tag{4.1}$$

satisfies (H_1') and (H_2') of Proposition 3.3 with Lipschitz constant K.

Notice that (H_7) holds if, for instance, b and $\partial_2 b$ are bounded continuous, σ satisfies (H_6), and $\partial_1\sigma/\sigma$ and $\partial_2\sigma/\sigma$ are bounded.

Finally, assume that

(H$_8$) χ is of class C^1 and $\chi' \leq \eta$, where

$$\begin{cases} \eta < \dfrac{e^{-K}}{M_0 M_1}, & \text{if } M_0 M_1 \neq \infty, \\ \eta = 0, & \text{if } M_0 M_1 = \infty, \end{cases}$$

with

$$M_0 := \sup_x \frac{1}{\sigma(0,x)} \quad \text{and} \quad M_1 := \sup_x \sigma(1,x).$$

Theorem 4.1. *Assume (H$_4$)–(H$_8$). Then the problem ($\mathcal{A}$) has a solution in* $\bigcap_p \mathsf{L}^{1,p}_{C,\text{loc}}$.

Proof. Let f be the function given in (H$_7$), and define $\psi: \mathbf{R} \to \mathbf{R}$ by

$$\psi(y) = G^{-1}(0, \chi(G(1,y))). \tag{4.2}$$

We want to apply the Itô formula of Theorem 2.1. First we must show that the solution Y of

$$\begin{cases} Y_t = Y_0 + \displaystyle\int_0^t f(s, Y_s)\, ds + W_t, & t \in [0,1], \\ Y_0 = \psi(Y_1), \end{cases}$$

belongs to $\mathsf{L}^{1,2}$. Actually, we are in the conditions of Proposition 3.6. Indeed, we know that f satisfies conditions (H$'_1$) and (H$'_2$), and, on the other hand,

$$\psi'(y) = \frac{\sigma(1, G(1,y))}{\sigma(0, \chi(G(1,y)))} \chi'(G(1,y)).$$

If $M_0 M_1 \neq \infty$, we get $\psi'(y) \leq M_0 M_1 \eta < \exp(-K)$, and if $M_0 M_1 = \infty$, we get $\psi'(y) \leq 0$. In both cases ψ satisfies (H$'_3$) and we conclude from Proposition 3.6 that $Y \in \bigcap_p \mathsf{L}^{1,p}_C$.

Now, it is easily seen that the processes $V^1_t = t$, $V^2_t = Y_t - W_t$ and $u \equiv 1$, and the function $\Phi(x_1, x_2; x_3) = G(x_1, x_2 + x_3)$ satisfy the assumptions of Theorem 2.1. Since

$$\begin{aligned} \Phi(V^1_t, V^2_t; U_t) &= G(t, Y_t), \\ \Phi(V^1_0, V^2_0; 0) &= G(0, Y_0), \\ \partial_1 \Phi(V^1_s, V^2_s; U_s) &= \partial_1 G(s, Y_s), \\ \partial_2 \Phi(V^1_s, V^2_s; U_s) &= \partial_3 \Phi(V^1_s, V^2_s; U_s) = \partial_2 G(s, Y_s), \\ dV^1_s &= ds, \\ dV^2_s &= f(s, Y_s) ds, \end{aligned}$$

we obtain

$$G(t,Y_t) = G(0,Y_0) + \int_0^t [\partial_1 G(s,Y_s) + \partial_2 G(s,Y_s) f(s,Y_s)]\, ds + \int_0^t \partial_2 G(s,Y_s) \circ dW_s,$$

that means, $X_t = G(t,Y_t)$ satisfies

$$X_t = X_0 + \int_0^t b(s,X_s)\, ds + \int_0^t \sigma(s,X_s) \circ dW_s.$$

On the other hand,

$$X_0 = G(0,Y_0) = G(0, G^{-1}(0, \chi(G(1,Y_1)))) = \chi(X_1),$$

and we have obtained a solution to $(\mathcal{A})$.

Let us see that $X \in \bigcap_p \mathsf{L}^{1,p}_{C,\mathrm{loc}}$. Denote $\Omega_n = \{\sup_{0\le t\le 1} |Y_t| \le n\}$, and let $\alpha_n\colon \mathbf{R} \to \mathbf{R}$ be a smooth function with $0 \le \alpha_n \le 1$, $\alpha_n(y) = 1$ on $\{|y| \le n\}$ and $\alpha_n(y) = 0$ on $\{|y| \ge n+1\}$. Define $X_t^{(n)} := G(t,Y_t)\alpha_n(Y_t)$, $n \ge 1$.

Clearly, $\Omega_n \nearrow \Omega$, and $X_t^{(n)} = X_t$ on Ω_n. By Lemma 2.2 (1), $X_t^{(n)} \in \mathsf{D}^{1,p}$ and

$$D_s X_t^{(n)} = \big[\sigma(t, G(t,Y_t))\alpha_n(Y_t) + G(t,Y_t)\alpha_n'(Y_t)\big] DsY_t.$$

We see that $X_t^{(n)}$ is uniformly bounded in t and that $|D_s X_t^{(n)}| \le R|D_s Y_t|$, for some constant R. Therefore, $X_t^{(n)} \in \mathsf{L}^{1,p}$. To conclude that $X \in \mathsf{L}^{1,p}_{C,\mathrm{loc}}$, we must check that the set of $L^p(\Omega)$-valued functions $\{s \mapsto D_{s\vee t} X^{(n)}_{s\wedge t}\}_{t\in[0,1]}$ is equicontinuous. This is tedious but straightforward, and we only sketch the proof. Using the bounds in (3.9), we can write

$$|D_{s_2\vee t} X^{(n)}_{s_2\wedge t} - D_{s_1\vee t} X^{(n)}_{s_1\wedge t}| \le C\big(|s_2 - s_1| + |Y_{s_2\wedge t} - Y_{s_1\wedge t}| + |s_2\wedge t - s_1 \wedge t|\big),$$

for some constant C. We have also, for $r \le t$,

$$|Y_t - Y_r| \le M|t-r| + |W_t - W_r| + K\int_r^t |Y_s|\, ds,$$

where $M = \sup_{0\le s\le 1} |f(s,0)|$. With the help of the inequality in Proposition 3.1 (2), we find

$$|Y_t(\omega)| \le C'(1 + \|\omega\|_\infty),$$

for some constant C', and from here we can finally deduce for any p the existence of a constant C'' such that

$$\mathsf{E}\left[|D_{s_2\vee t} X^{(n)}_{s_2\wedge t} - D_{s_1\vee t} X^{(n)}_{s_1\wedge t}|^p\right] \le C''|s_2 - s_1|^p.$$

Analogously, one obtains the equicontinuity of $\{s \mapsto D_{s\wedge t} X^{(n)}_{s\vee t}\}_{t\in[0,1]}$, and this finishes the proof of the theorem. □

Remark 4.1. In case σ is a bounded function, it is not difficult to see that the process X found in Theorem 4.1 actually belongs to the spaces $\mathsf{L}^{1,p}$ (there is no need to localize). If moreover $\partial_1\sigma$ and $\partial_2\sigma$ are bounded, then X belongs to $\mathsf{L}^{1,p}_C$. In any case, by the chain rule (Lemma 2.2 (1)) we get $D_sX_t = \sigma(t, X_t)D_sY_t$. It is also immediate that X is a Fréchet differentiable mapping on Ω and $\nabla X_t(\omega) = \sigma(t, X_t(\omega))\nabla Y_t(\omega)$, for all ω.

Using the inverse change of variables $Y_t = G^{-1}(t, X_t)$, one can formally transform a solution X of equation $(\mathcal{A})$ into a solution Y of an equation of type $(\mathcal{B})$. Therefore, if we assume that X belongs to a class of processes for which Theorem 2.1 can be applied to

$$V_t^1 = t, \quad V_t^2 = X_0 + \int_0^t b(s, X_s)\, ds, \quad u_t = \sigma(t, X_t),$$

and

$$\Phi(x_1, x_2; x_3) = G^{-1}(x_1, x_2 + x_3),$$

then the uniqueness property of equation $(\mathcal{B})$ implies at once the uniqueness for $(\mathcal{A})$ inside the given class of processes.

We will not state here a rigorous result in this direction. Some uniqueness theorems for stochastic differential equations with boundary conditions can be found in [1, 5, 9].

5. Markovian-type properties

Definition 5.1. We say that $X = \{X_t,\ t \in [0,1]\}$ is a *reciprocal process* if for every $0 < a < b < 1$, $\{X_t,\ t \in [a,b]\}$ and $\{X_t,\ t \in [0,1] -]a,b[\}$ are conditionally independent given $\{X_a, X_b\}$.

A reciprocal process (a notion introduced by Bernstein [2]) is a one-dimensional *Markov field* in Paul Lévy's terminology. The concept can also be found in the literature under the names of Bernstein process, quasi-Markov process, and local Markov process. Note that the Markov field property in the sense of Rozanov [14] is weaker. To distinguish between both concepts, sometimes the former is called *sharp Markov field,* and the latter *germ Markov field.*

Jamison [7] shows that every continuous Markov process is reciprocal. The converse is false: the process $X_t = W_t - 1/2W_1$, $t \in [0,1]$, where W is a Wiener process, is reciprocal, but not Markovian. It is the solution to the trivial boundary value problem

$$\begin{cases} X_t = X_0 + W_t, & t \in [0,1], \\ X_0 = -X_1. \end{cases}$$

The following result is an immediate consequence of Theorem 4.1 in [1], and characterizes the set of equations of type $(\mathcal{B})$ for which the reciprocal property holds true.

Theorem 5.1. *Assume* $f: [0,1] \times \mathbf{R} \to \mathbf{R}$ *and* $\psi: \mathbf{R} \to \mathbf{R}$ *satisfy* (H_1'), (H_2') *and* (H_3'). *Then the solution to* $(\mathcal{B})$ *is a reciprocal process if and only if one of the following conditions holds:*

a) $\psi' \equiv 0$.

b) $f(t,x) = \alpha(t)x + \beta(t)$, *for some functions* α *and* β.

Now, using this theorem and the representation $X_t = G(t, Y_t)$ for the solution to $(\mathcal{A})$, we can find the necessary and sufficient conditions on b, σ and χ under which X_t is a reciprocal process. This generalizes Theorem 5.2 of [1].

Theorem 5.2. *Under the assumptions of Theorem 4.1, the solution to the problem* $(\mathcal{A})$ *found there is a reciprocal process if and only if one of the following conditions holds:*

(a) $\chi' \equiv 0$.

(b) $b(t,x) = \sigma(t,x)\Big[\dfrac{b(t,0)}{\sigma(t,0)} + a(t)\displaystyle\int_0^x \frac{1}{\sigma(t,s)}\,ds + \int_0^x \frac{\partial_1\sigma(t,s)}{\sigma^2(t,s)}\,ds\Big]$, *for some function* $a(t)$.

Proof. First observe that, since $X_t = G(t, Y_t)$ if and only if $Y_t = G^{-1}(t, X_t)$, the σ-fields generated by X_t and by Y_t coincide, for each t. Therefore, X is a reciprocal process if and only if Y is. According to Theorem 5.1, this is true if and only if $\psi' \equiv 0$ or $f(t,x) = \alpha(t)x + \beta(t)$.

From (4.2) and the fact that G is strictly increasing, we see that $\psi' \equiv 0$ is equivalent to the condition $\chi' \equiv 0$. On the other hand, differentiating (4.1) with respect to the second argument, we find that the second possibility is equivalent to the existence of a function $a(t)$ such that

$$\partial_2 b(t,x) - \frac{\partial_1\sigma(t,x)}{\sigma(t,x)} - \frac{\partial_2\sigma(t,x)}{\sigma(t,x)} b(t,x) = a(t). \tag{5.1}$$

Solving for b one obtains

$$\begin{aligned} b(t,x) &= b(t,0)\exp\Big\{\int_0^x \frac{\partial_2\sigma(t,\xi)}{\sigma(t,\xi)}\,d\xi\Big\} \\ &\quad + \int_0^x \Big[a(t) + \frac{\partial_1\sigma(t,s)}{\sigma(t,s)}\Big]\exp\Big\{\int_s^x \frac{\partial_2\sigma(t,\xi)}{\sigma(t,\xi)}\,d\xi\Big\}\,ds \\ &= \sigma(t,x)\Big[\frac{b(t,0)}{\sigma(t,0)} + a(t)\int_0^x \frac{1}{\sigma(t,s)}\,ds + \int_0^x \frac{\partial_1\sigma(t,s)}{\sigma^2(t,s)}\,ds\Big]. \end{aligned}$$

□

Corollary 5.1. *Consider the boundary value problem*

$$\begin{cases} X_t = X_0 + \int_0^t \sigma(s, X_s) \circ dW_s, & t \in [0,1], \\ X_0 = \chi(X_1), \end{cases}$$

and assume (H_4)–(H_8) *(with* $b \equiv 0$*).*

Then X *is a reciprocal process if and only if* $\chi' \equiv 0$ *or* σ *factorizes as* $\sigma(s,x) = A(s)B(x)$. *In particular, if* σ *does not depend on* s, X *is always a reciprocal process.*

Proof. When $b \equiv 0$, condition (5.1) reduces to

$$\frac{\partial_1 \sigma(t,x)}{\sigma(t,x)} = -a(t),$$

hence

$$\sigma(s,x) = \sigma(0,x) \exp\Big\{ - \int_0^s a(t)\, ds \Big\} = A(s)B(x).$$

□

6. The linear case

In this section we consider the case in which σ is linear in the second variable, that means $\sigma(t,x) = \sigma(t)x$, with $\sigma(t) > 0$ for all $t \in [0,1]$. This case is not covered by the results of Sections 4 and 5, because the diffusion coefficient vanishes at the points $(t,0)$. However, under a certain assumption on the boundary condition, a solution exists which avoids the singular point and can be constructed, with the same change of variable technique that we have used before, from the solution Y of an equation with unit diffusion. We have the following theorem.

Theorem 6.1. *Let* $\sigma: [0,1] \to \mathbf{R}$, $b: [0,1] \times \mathbf{R} \to \mathbf{R}$ *and* $\chi: \mathbf{R} \to \mathbf{R}$ *be functions satisfying the following conditions.*

a) $\sigma \in C^1$ *and* $\sigma > 0$.

b) *The function* $f: [0,1] \times \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(t,y) = \frac{b(t, e^{\sigma(t)y})}{\sigma(t) e^{\sigma(t)y}} - \frac{\sigma'(t)}{\sigma(t)} y \tag{6.1}$$

satisfies assumptions (H_1') *and* (H_2'), *with Lipschitz constant* K.

c) $\chi \in C^1$, *maps positive numbers into positive numbers, and*

$$\frac{\sigma(1)\chi'(e^{\sigma(1)y})}{\sigma(0)\chi(e^{\sigma(1)y})} e^{\sigma(1)y} \leq \eta \tag{6.2}$$

for some $\eta < \exp(-K)$.

Then, the problem

$$(\mathcal{L}) \quad \begin{cases} X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s) X_s \circ dW_s, & t \in [0,1], \\ X_0 = \chi(X_1) \end{cases}$$

has a solution in $\bigcap_p \mathbb{L}^{1,p}_{\text{loc}}$, *which moreover has positive paths.*

Proof. Define

$$\psi(y) := \frac{1}{\sigma(0)} \log\left[\chi(e^{\sigma(1)y})\right].$$

This function satisfies the assumption (H_3') of Proposition 3.5, because $\psi'(y)$ is equal to the left-hand side in (6.2). This ensures that

$$\begin{cases} Y_t = Y_0 + \int_0^t f(s, Y_s)\, ds + dW_t, & t \in [0,1], \\ Y_0 = \psi(Y_1) \end{cases}$$

has a unique solution, which belongs to $\bigcap_p \mathbb{L}^{1,p}_C$. Applying Theorem 2.1 to the processes $V_t^1 = t$, $V_t^2 = Y_t - W_t$, $u \equiv 1$, and the function $\Phi(x_1, x_2; x_3) = \exp(\sigma(x_1)(x_2 + x_3))$, the continuous process defined by $X_t := \exp(\sigma(t) Y_t)$ verifies

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s) X_s \circ dW_s$$

and

$$X_0 = e^{\sigma(0) Y_0} = e^{\sigma(0)\psi(Y_1)} = \chi(e^{\sigma(1) Y_1}) = \chi(X_1).$$

Proceeding as in the proof of Theorem 4.1, one can see that $X_t \in \bigcap_p \mathbb{L}^{1,p}_{\text{loc}}$. □

Notice that from the results of Section 5, X_t is a reciprocal process if and only if

a) $\chi' \equiv 0$,

or

b) $b(t,x) = b(t,1)x + \left[\frac{\sigma'(t)}{\sigma(t)} + a(t)\right] x \log x$, for all $x > 0$ and some function $a(t)$.

Theorem 6.1 is also true if we assume that χ applies negative numbers into negative numbers. In this case, we replace $\exp(\sigma(t)y)$ by $-\exp(\sigma(t)y)$ in (b) and $\exp(\sigma(1)y)$ by $-\exp(\sigma(1)y)$ in (c), and we obtain a negative solution $X_t = -\exp(\sigma(t)Y_t)$. Condition (b) for the reciprocal property is then

$$b(t,x) = -b(t,1)x + \left[\frac{\sigma'(t)}{\sigma(t)} + a(t)\right] x \log(-x), \quad \forall x < 0.$$

Particularizing a little more, we assume now that the boundary condition is also linear: $X_0 = \alpha X_1 + \beta$. In case $\alpha \geq 0$, this situation is covered by the previous theorem (in its "positive" or "negative" version, depending on $\beta \geq 0$, or $\beta \leq 0$, respectively).

Assume that $\alpha < 0$ and $\beta > 0$. Consider the auxiliary problem

$$(\mathcal{L})' \quad \begin{cases} Y_t = Y_0 + \int_0^t f(s, Y_s)\, ds + W_t, & t \in [0,1], \\ Y_0 = \frac{1}{\sigma(0)} \log\left(\alpha e^{\sigma(1)Y_1} + \beta\right), \end{cases}$$

with f given by (6.1). The boundary condition can be written as

$$Y_1 = g(Y_0) := \frac{1}{\sigma(1)} \log\left[\frac{-1}{\alpha}\left(\beta - e^{\sigma(0)Y_0}\right)\right].$$

The function g is a decreasing bijection between the intervals

$$\left]-\infty, \frac{1}{\sigma(0)} \log \beta\right[\quad \text{and} \quad \left]-\infty, \frac{1}{\sigma(1)} \log \frac{-\beta}{\alpha}\right[.$$

Since $\varphi_1'(\omega, y) \geq \exp(-K) > 0$ (see (3.2)), equation $\varphi_1(\omega, y) - g(y) = 0$ will have a unique solution, that we call Y_0. Then the process $Y_t = \varphi_t(\omega, Y_0(\omega))$ is the unique solution to $(\mathcal{L})'$, and, as before, it can be seen that it belongs to the spaces $\mathbb{L}_C^{1,p}$. Now we apply the change of variables $X_t = \exp(\sigma(t)Y_t)$ as in Theorem 6.1, and we get a solution to our equation.

Finally, the case $X_0 = \alpha X_1 + \beta$ with $\alpha < 0$ and $\beta < 0$ can be treated in a similar way, with the change of variables $X_t = -\exp(\sigma(t)Y_t)$, and writing $Y_0 = (1/\sigma(0)) \log\left(\alpha \exp(\sigma(1)Y_1) - \beta\right)$ in $(\mathcal{L})'$.

It is clear that our method can only work to construct a solution of $(\mathcal{L})$ whose paths are all of them positive or all of them negative. For example, the equation

$$\begin{cases} X_t = X_0 + \int_0^t X_s \circ dW_s, & t \in [0,1], \\ X_0 = X_1 + 1, \end{cases}$$

can be explicitly solved and the solution has positive and negative paths. Theorem 6.1 cannot be applied here because the constant η in (c) does not exist.

7. Other properties of the solution

The representation $X_t = G(t, Y_t)$ for the solution to $(\mathcal{A})$, where Y_t is the solution to the simpler equation $(\mathcal{B})$, allows to study some properties of the former quite easily, as we have seen with the Markovian property of Section 5. Here we will show also the absolute continuity of the law of X_t, a maximal inequality and a comparison result.

Proposition 7.1. *Let $\sigma, b\colon [0,1] \times \mathbf{R} \to \mathbf{R}$ and $\chi\colon \mathbf{R} \to \mathbf{R}$ satisfy assumptions* (H_4)–(H_8) *and let $X = \{X_t,\ t \in [0,1]\}$ be the solution to $(\mathcal{A})$ found in Theorem* 4.1.

Then, for every $t \in (0,1]$, the law of X_t is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$. If $\chi' \neq 0$ a.e., this is also true for X_0.

Proof. We know that $\|D_s X_t\|_H = \sigma(t, X_t)\|D_s Y_t\|_H$. By (3) of Lemma 2.2, it is enough to show that the last norm is positive.

From (3.6), multiplying $D_s Y_t$ by $\exp\{-\int_s^t \partial_2 f(r, Y_r(\omega))\,dr\} > 0$, we see that $\|D_s Y_t\|_H = 0$ if and only if the function of s

$$\frac{\psi'(\varphi_1(\omega, Y_0(\omega)))}{1 - \psi'(\varphi_1(\omega, Y_0(\omega)))\varphi_1'(\omega, Y_0(\omega))} \exp\Big\{\int_0^1 \partial_2 f(r, \varphi_r(\omega, Y_0(\omega)))\,dr\Big\} + \mathbf{1}_{[0,t]}(s)$$

is zero a.e.

If $0 < t < 1$, this is clearly impossible. If $t = 1$, using

$$\varphi_1'(\omega, Y_0(\omega)) = \exp\Big\{\int_0^1 \partial_2 f(r, \varphi_r(\omega, Y_0(\omega)))\,dr\Big\},$$

we get also an absurd equality.

With the same arguments of Proposition 3.1 in [1], one can show that the support of the law of Y_1 is the whole real line. Therefore, we see from (3.7) that Y_0 (and consequently X_0) is absolutely continuous provided $\varphi' \neq 0$ a.e. (equivalently $\chi' \neq 0$ a.e.). □

Proposition 7.2. *Assume $\sigma, b\colon [0,1] \times \mathbf{R} \to \mathbf{R}$ and $\chi\colon \mathbf{R} \to \mathbf{R}$ satisfy assumptions* (H_4)–(H_8) *and let $X = \{X_t,\ t \in [0,1]\}$ be the solution to $(\mathcal{A})$ found in Theorem* 4.1.

Then, for all $p > 1$,

$$\mathsf{E}\Big[\sup_{0\le t\le 1} |X_t|^p\Big] \le R^p 3^{(p-1)} e^{pK}\Big[M^p + D^p + (C+1)^p\Big(\frac{p}{p-1}\Big)^p \mathsf{E}(|W_1|^p)\Big],$$

where K is the constant in (H_7),

$$R = \sup_{t,x} \sigma(t,x), \quad M = \sup_t \left|\frac{b(t,0)}{\sigma(t,0)}\right|, \quad D = |G^{-1}(0, X_0(0))|,$$

and

$$C = \max\{e^{2K}, \frac{1}{1 - M_0 M_1 \eta e^K}\}.$$

In particular, if σ is bounded, then $\sup_{0\le t\le 1} |X_t| \in \bigcap_p L^p(\Omega)$.

Proof. We have $|X_t| = |G(t,Y_t)| \le R|Y_t|$, where Y is the solution to problem $(\mathcal{B})$. By Proposition 3.1 (1),

$$|Y_t(\omega)| = |\varphi_t(\omega, Y_0(\omega))| \le \big[|Y_0(\omega)| + \|\omega\|_\infty + M\big]e^K.$$

We know from the proof of Proposition 3.6 that $|Y_0(\omega)| \le |Y_0(0)| + C\|\omega\|_\infty = D + C\|\omega\|_\infty$. Hence,

$$|Y_t(\omega)|^p \le 3^{(p-1)} e^{pK}\big[D^p + M^p + (C+1)^p \|\omega\|_\infty^p\big].$$

The Doob maximal inequality states that

$$\mathsf{E}[(\sup_{0\le s\le 1} |W_s|)^p] \le (\frac{p}{p-1})^p\, \mathsf{E}[|W_1|^p],$$

and the result follows. □

We finish with an easy comparison result with respect to the boundary function χ.

Proposition 7.3. *Let σ, b: $[0,1] \times \mathbf{R} \to \mathbf{R}$ and χ: $\mathbf{R} \to \mathbf{R}$ satisfy assumptions* (H_4)–(H_8) *and let $X = \{X_t,\ t \in [0,1]\}$ be the solution to $(\mathcal{A})$ found in Theorem 4.1.*

Let $\bar\chi$: $\mathbf{R} \to \mathbf{R}$ be another function satisfying (H_8), *and such that $\chi \le \bar\chi$. Let $\bar X = \{\bar X_t,\ t \in [0,1]\}$ be the corresponding solution to*

$$\begin{cases} \bar X_t = \bar X_0 + \int_0^t b(s, \bar X_s)\, ds + \int_0^t \sigma(s, \bar X_s) \circ dW_s, \quad t \in [0,1], \\ \bar X_0 = \bar\chi(\bar X_1). \end{cases}$$

Then $X_t \le \bar X_t$.

Proof. Define

$$\psi(y) := G^{-1}(0, \chi(G(1,y))) \quad \text{and} \quad \bar\psi(y) := G^{-1}(0, \bar\chi(G(1,y))).$$

Since $G^{-1}(0,\cdot)$ is increasing, we have $\psi \leq \bar{\psi}$. If $\bar{Y} = \{\bar{Y}_t,\ t \in [0,1]\}$ solves

$$\begin{cases} \bar{Y}_t = \bar{Y}_0 + \int_0^t f(s,\bar{Y}_s)\,ds + W_t, & t \in [0,1], \\ \bar{Y}_0 = \bar{\psi}(\bar{Y}_1), \end{cases}$$

then

$$Y_0(\omega) - \bar{\psi}(\varphi_1(\omega, Y_0(\omega))) \leq Y_0(\omega) - \psi(\varphi_1(\omega, Y_0(\omega))) = 0,$$

which implies that $Y_0(\omega) \leq \bar{Y}_0(\omega)$. By Proposition 3.2 (3),

$$Y_t(\omega) = \varphi_t(\omega, Y_0(\omega)) \leq \varphi_t(\omega, \bar{Y}_0(\omega)) = \bar{Y}_t(\omega).$$

Finally,

$$X_t = G(t, Y_t) \leq G(t, \bar{Y}_t) = \bar{X}_t.$$

□

References

[1] A. ALABERT, M. FERRANTE AND D. NUALART (1995) Markov field property of stochastic differential equations. *Ann. Prob.* **23**, 1262–1288.

[2] S. BERNSTEIN (1932) Sur les liaisons entre les grandeurs aléatoires. *Proc. Int. Cong. Math.*, Zurich, 288–309.

[3] H. CARTAN (1967) *Calcul Différentiel.* Hermann, Paris.

[4] C. DONATI-MARTIN (1991) Equations différentielles stochastiques dans R avec conditions aux bords. *Stochastics and Stochast. Reports* **35**, 143–173.

[5] J. GARNIER (1995) Stochastic invariant imbedding. Application to stochastic differential equations with boundary conditions. *Probab. Theory and Relat. Fields* **102**, 249–271.

[6] I.I. GIHMAN AND A.V. SKOROHOD (1972) *Stochastic Differential Equations.* Springer-Verlag.

[7] B. JAMISON (1970) Reciprocal processes: the stationary gaussian case. *Ann. Math. Stat.* **41** (5), 1624–1630.

[8] P.E. KLOEDEN AND E. PLATEN (1992) *Numerical Solution of Stochastic Differential Equations.* Springer-Verlag.

[9] A. KOHATSU-HIGA, J.A. LEÓN AND D. NUALART (1997) Stochastic differential equations with random coefficients. *Bernoulli* **3**, 233–245.

[10] N.V. NORIN (1996) *The Extended Stochastic Integral in Linear Spaces with Differentiable Measures and Related Topics.* World Scientific Publishing Co. Pte. Ltd.

[11] D. NUALART (1995) *The Malliavin Calculus and Related Topics.* Springer-Verlag.

[12] D. NUALART AND E. PARDOUX (1988) Stochastic calculus with anticipating integrands. *Probab. Theory and Relat. Fields* **78**, 535–581.

[13] D. Ocone and E. Pardoux (1989) Linear stochastic differential equations with boundary conditions. *Probab. Theory and Relat. Fields* **82**, 489–526.

[14] Yu.A. Rozanov (1982) *Markov Random Fields.* Springer-Verlag.

[15] H. Sugita (1985) On a characterization of the Sobolev spaces over an abstract Wiener space. *J. Math. Kyoto Univ.* **25** (4), 717–725.