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Differential equations with boundary conditions perturbed by a Poisson noise

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Abstract

We consider one-dimensional differential equations with a boundary condition on the interval $[0,1]$, perturbed by a Poisson noise. We study existence and uniqueness, the law of the solution and in which cases the solution is a reciprocal process. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the last years some papers have been written on stochastic differential equations with boundary conditions driven by a white noise, namely, problems of the form

$$\begin{aligned} X_t &= X_0 + \int_0^t f(s,X_s)\,\mathrm{d}s + \int_0^t \sigma(s,X_s)\circ \mathrm{d}W_s, \quad t\in[0,1]. \\ X_0 &= \psi(X_1). \end{aligned} \tag{1.1}$$

Equations of the form (1.1) are anticipating in nature, and they have provided a field of applications for the anticipating stochastic calculus developed in the late 1980s. We refer the reader interested in these white noise driven equations to the survey (Alabert, 1995), the references therein, and the latter papers (Alabert et al., 1995; Ferrante and Nualart, 1992; Alabert and Marmolejo, 1999).

The solution to (1.1) is not a Markov process, except in trivial cases, due to the strong relationship between the variables X_0 and X_1. Instead, investigation of conditional independence properties of the solution has turned mainly to the weaker *reciprocal*

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property (see Definition 4.1), also called Markov-field, quasi-Markov or local-Markov property in the literature. Reciprocal processes have received considerable attention recently (see, e.g., Krener, 1997), since they play an important role in the efforts to find a satisfactory stochastic theory of quantum mechanics.

So far, nothing has been written concerning boundary value problems driven by a Poisson noise. In this paper we provide a first work in this direction. We consider the simplest possible case, in which the noise appears additively:

$$\begin{aligned} X_t &= X_0 + \int_0^t f(s, X_s)\,\mathrm{d}s + N_t, \quad t \in [0,1]. \\ X_0 &= \psi(X_1). \end{aligned} \tag{1.2}$$

The solution will be taken in a pathwise sense, and, therefore, we do not need to use any kind of anticipating stochastic calculus with respect to the Poisson process. The case with multiplicative noise is a work currently under development.

In the Wiener setting, two methods have been employed to study Markovian-type properties of the solution. One of them is based in an anticipating change of measure in Wiener space (the Ramer–Kusuoka Theorem, see Kusuoka, 1982); the second one relies on a more direct argument that involves a characterisation of the conditional independence of two independent random vectors given a function of them, and was introduced in Alabert et al. (1992) (see Lemma 4.3 below). In our case, the first method cannot be applied; our approach is based on the second method, which, in comparison with the Wiener case, has some additional technical difficulties, arising from several facts:

(1) The laws of the random variables X_t have in general a discrete and an absolutely continuous part.
(2) The support of these laws is not the whole real line.
(3) The path space of the Poisson process is not as rich as the Wiener space.

The results obtained here differ from the ones we know for white noise driven equations. Indeed, it was shown in Nualart and Pardoux (1991) that, when $\sigma \equiv 1$, the solution to (1.1) is a reciprocal process if and only if f is an affine function in the second variable (or ψ is constant); for our Eq. (1.2), we find that if f is affine or 1-periodic in the second variable then the solution is reciprocal (Theorems 4.7 and 4.8), but the converse is not true (Example 4.9). These results are proved by direct arguments. As Example 4.9 suggests, it seems difficult to find a neat necessary and sufficient condition for the reciprocal property to hold. However, in the autonomous case, and by means of the aforementioned characterisation of the conditional independence, we are able to identify a wide class of drift coefficients for which the reciprocal property fails (Theorem 4.10). This class includes the functions $f > 0$ for which

$$F(x) = \frac{f'(x+1)f(x) - f(x+1)f'(x)}{f(x+1) - f(x)}$$

is well-defined and one-to-one.

The process N in (1.2) models an impulse disturbance independent of time and position. It is also interesting to study the case when the source of the disturbance is finite, modelled by a Poisson process conditioned to have no more than a fixed number of jumps in the interval $[0,1]$. This situation is more difficult to deal with, mainly because the increments of the driving process are not independent. We will concentrate in the case when at most one jump is allowed. It is shown that the solution is always reciprocal (Theorem 5.1), and it is also Markov under some conditions (Theorem 5.2).

We explain now briefly what can be found in each of the sections of this work. Section 2 is devoted to some preliminaries about the equation $\mathrm{d}X_t = f(t,X_t) + \mathrm{d}N_t$ with constant initial condition. In Section 3 we study the existence and uniqueness of a solution to (1.2), its differentiability properties when the coefficients are smooth and the absolute continuity of the law of each variable X_t with respect to the sum of a Dirac-δ measure and the Lebesgue measure. In Section 4 we state and prove the main results of this paper, which deal with the reciprocal property of the solution. Finally, in Section 5, we give some results on the Markov and reciprocal properties with a finite source of Poissonian noise.

We will use the notation $\partial_i f$ for the derivative of a function f with respect to the ith coordinate, $f(s^-)$ and $f(s^+)$ for $\lim_{t\uparrow s} f(t)$ and $\lim_{t\downarrow s} f(t)$, respectively, and the acronym *càdlàg* for "right continuous with left limits".

2. The equation of the flow

Let $N = \{N_t,\ t\geqslant 0\}$ be a standard Poisson process with intensity 1 defined on some probability space $(\Omega, \mathfrak{F}, P)$; that means, N has independent increments, $N_t - N_s$ has a Poisson law with parameter $t-s$, $N_0 \equiv 0$, and all its paths are integer-valued, non-decreasing, càdlàg, with jumps of size 1.

Throughout the paper, S_n will denote the jump times of N:

$$S_n(\omega) := \inf\{t\geqslant 0 : N_t(\omega)\geqslant n\}.$$

The sequence S_n is strictly increasing to infinity, and $\{N_t = n\} = \{S_n \leqslant t < S_{n+1}\}$.

Let us consider the pathwise equation

$$\varphi_{st}(x) = x + \int_s^t f(r, \varphi_{sr}(x))\,\mathrm{d}r + N_t - N_s, \quad 0\leqslant s\leqslant t\leqslant 1, \tag{2.1}$$

where $x\in\mathbb{R}$, and assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that
(H_1) $\exists K>0$: $\forall t\in[0,1]$, $\forall x,y\in\mathbb{R}$, $|f(t,x)-f(t,y)|\leqslant K|x-y|$,
(H_2) $M := \sup_{t\in[0,1]} |f(t,0)| < \infty$.

For every $x\in\mathbb{R}$, denote by $\Phi(s,t;x)$ the solution to the deterministic equation

$$\Phi(s,t;x) = x + \int_s^t f(r, \Phi(s,r;x))\,\mathrm{d}r, \quad 0\leqslant s\leqslant t\leqslant 1. \tag{2.2}$$

The following properties of (2.2) are either well known to the reader or very easy to show:

Proposition 2.1. *Under hypotheses* (H_1) *and* (H_2), *there exists a unique solution* $\Phi(s,t;x)$ *of Eq.* (2.2). *Moreover,*
(1) *For every* $0\leqslant s\leqslant t\leqslant 1$, *and every* $x\in\mathbb{R}$, $|\Phi(s,t;x)|\leqslant(|x|+M)\mathrm{e}^{K(t-s)}$.
(2) *For every* $0\leqslant s\leqslant r\leqslant t\leqslant 1$, *and every* $x\in\mathbb{R}$, $\Phi(r,t;\Phi(s,r;x)) = \Phi(s,t;x)$.

(3) *For every* $0 \leqslant s \leqslant t \leqslant 1$, *and every* $x_1, x_2 \in \mathbb{R}$ *with* $x_1 < x_2$,

$$(x_2 - x_1)\mathrm{e}^{-K(t-s)} \leqslant \Phi(s,t;x_2) - \Phi(s,t;x_1) \leqslant (x_2 - x_1)\mathrm{e}^{K(t-s)}.$$

In particular, for every s,t, *the function* $x \mapsto \Phi(s,t;x)$ *is a homeomorphism from* $\mathbb{R}$ *into* $\mathbb{R}$.

(4) *If* $G:[0,1] \times \mathbb{R} \to \mathbb{R}$ *has continuous partial derivatives, then for every* $0 \leqslant s \leqslant t \leqslant 1$,

$$G(t, \Phi(s,t;x)) = G(s,x) + \int_s^t [\partial_1 G(r, \Phi(s,r;x)) + \partial_2 G(r, \Phi(s,r;x)) f(r, \Phi(s,r;x))]\,\mathrm{d}r.$$

Using Proposition 2.1 one can prove easily the following analogous properties for Eq. (2.1):

Corollary 2.2. *Under hypotheses* (H_1) *and* (H_2), *for every* $x \in \mathbb{R}$ *there is a unique process* $\varphi(x) = \{\varphi_{st}(x),\ 0 \leqslant s \leqslant t \leqslant 1\}$ *that solves* (2.1). *Moreover*,

(1) *For every* $0 \leqslant s \leqslant t \leqslant 1$, *and every* $x \in \mathbb{R}$,

$$|\varphi_{st}(x)| \leqslant (|x| + (N_t - N_s)(1 + M) + M)\mathrm{e}^{K(t-s)}.$$

(2) *For every* $0 \leqslant s \leqslant r \leqslant t \leqslant 1$, *and every* $x \in \mathbb{R}$, $\varphi_{rt}(\varphi_{sr}(x)) = \varphi_{st}(x)$.

(3) *For every* $0 \leqslant s \leqslant t \leqslant 1$, *and every* $x_1, x_2 \in \mathbb{R}$ *with* $x_1 < x_2$,

$$(x_2 - x_1)\mathrm{e}^{-K(t-s)} \leqslant \varphi_{st}(x_2) - \varphi_{st}(x_1) \leqslant (x_2 - x_1)\mathrm{e}^{K(t-s)}.$$

In particular, for every s,t, *the function* $x \mapsto \varphi_{st}(x)$ *is a random homeomorphism from* $\mathbb{R}$ *into* $\mathbb{R}$.

(4) *If* $G:[0,1] \times \mathbb{R} \to \mathbb{R}$ *has continuous partial derivatives, then for every* $0 \leqslant s \leqslant t \leqslant 1$,

$$G(t, \varphi_{st}(x)) = G(s,x) + \int_s^t [\partial_1 G(r, \varphi_{sr}(x)) + \partial_2 G(r, \varphi_{sr}(x)) f(r, \varphi_{sr}(x))]\,\mathrm{d}r + \int_{s^+}^t [G(r, \varphi_{sr}(x)) - G(r, \varphi_{sr^-}(x))]\,\mathrm{d}N_r.$$

Notice that the jumps of φ coincide with those of N in position and size, and that the homeomorphism property above is not true for equations driven by general martingales with jumps (see Meyer, 1981 or Léandre, 1985).

By solving Eq. (2.2) between jumps, the value $\varphi_{st}(\omega, x)$ can be found recursively in terms of Φ: If $s_1 = S_1(\omega), \ldots, s_n = S_n(\omega)$ are the jump times of the path $N(\omega)$ on $(s,1]$, then

$$\varphi_{st}(x) = \Phi(s,t;x)\mathbf{1}_{[s,s_1)}(t) + \sum_{i=1}^{n-1} \Phi(s_i, t; \varphi_{ss_i^-}(x) + 1)\mathbf{1}_{[s_i, s_{i+1})}(t) + \Phi(s_n, t; \varphi_{ss_n^-}(x) + 1)\mathbf{1}_{[s_n, 1]}(t). \tag{2.3}$$

Consider now the stronger hypotheses

(H_1') f and $\partial_2 f$ are continuous functions.
(H_2') $\exists K > 0$: $|\partial_2 f| \leqslant K$.

Proposition 2.3. *Under hypotheses* (H_1') *and* (H_2'), *we have*:

(1) *For every* $\omega \in \Omega$ *and every* $x \in \mathbb{R}$, *the function* $t \mapsto \varphi_{st}(\omega, x)$ *is differentiable on* $[s,1] - \{s_1, s_2, \ldots\}$, *where* $s_1, s_2, \ldots$ *are the jump times of* $N(\omega)$ *on* $(s,1]$, *and*

$$\frac{\mathrm{d}\varphi_{st}(x)}{\mathrm{d}t} = f(t, \varphi_{st}(x)).$$

(2) *For every* $\omega \in \Omega$ *and every* $0 \leqslant s < t \leqslant 1$, *the function* $x \mapsto \varphi_{st}(\omega, x)$ *is continuously differentiable and*

$$\frac{\mathrm{d}\varphi_{st}(x)}{\mathrm{d}x} = \exp\left\{\int_s^t \partial_2 f(r, \varphi_{sr}(x))\,\mathrm{d}r\right\}.$$

In particular, $x \mapsto \varphi_{st}(x)$ *is a random diffeomorphism from* $\mathbb{R}$ *into* $\mathbb{R}$.

(3) *On the set* $\{N_t - N_s = n\}$, $(n = 1, 2, \ldots)$, *the mapping* $\omega \mapsto \varphi_{st}(\omega, x)$ *can be regarded as a function* $\varphi_{st}(s_1, \ldots, s_n; x)$ *defined on* $\{s < s_1 < \cdots < s_n \leqslant t\}$, *where* $s_j = S_j(\omega)$ *are the jump times of* $N(\omega)$ *in* $(s,t]$. *This function is continuously differentiable and, for every* $j \in \{1, \ldots, n\}$,

$$\frac{\partial\varphi_{st}(x)}{ds_j} = \exp\left\{\int_{s_j}^t \partial_2 f(r, \varphi_{sr}(x))\,\mathrm{d}r\right\}[f(s_j, \varphi_{ss_j^-}(x)) - f(s_j, \varphi_{ss_j}(x))].$$

Proof. It is easy to see that

$$\partial_1 \Phi(s,t;x) = -f(s,x)\exp\left\{\int_s^t \partial_2 f(r, \Phi(s,r;x))\,\mathrm{d}r\right\},$$

$$\partial_2 \Phi(s,t;x) = f(t, \Phi(s,t;x)),$$

$$\partial_3 \Phi(s,t;x) = \exp\left\{\int_s^t \partial_2 f(r, \Phi(s,r;x))\,\mathrm{d}r\right\}$$

and that these derivatives are continuous on $\{0 \leqslant s \leqslant t \leqslant 1\} \times \mathbb{R}$. Claims (1) and (2) follow from these formulae and representation (2.3).

The existence of the function $\varphi_{st}(s_1, \ldots, s_n; x)$ of (3) and its differentiability properties are also clear from (2.3). Let us compute the derivative with respect to s_j. For $n = 1$, we have

$$\frac{\mathrm{d}\varphi_{st}(x)}{\mathrm{d}s_1} = \partial_1 \Phi(s_1, t; \Phi(s, s_1; x) + 1) + \partial_3 \Phi(s_1, t; \Phi(s, s_1; x) + 1)\partial_2 \Phi(s, s_1; x)$$

$$= \exp\left\{\int_{s_1}^t \partial_2 f(r, \varphi_{sr}(x))\right\}\left[f(s_1, \varphi_{ss_1^-}(x)) - f(s_1, \varphi_{ss_1}(x))\right].$$

Assume the formula is valid up to $n = k$. Then, for $n = k + 1$, and $j = 1,\dots,k$,

$$\frac{\partial\varphi_{st}(x)}{\partial s_j} = \partial_3\Phi(s_{k+1},t;\varphi_{ss_{k+1}}(x))\,\frac{\partial\varphi_{ss_{k+1}}(x)}{\partial s_j}$$

$$= \exp\left\{\int_{s_j}^{t} \partial_2 f(r,\varphi_{sr}(x))\,\mathrm{d}r\right\}\left[f(s_j,\varphi_{ss_j^-}(x)) - f(s_j,\varphi_{ss_j}(x))\right].$$

Taking into account that

$$\varphi_{ss_{k+1}}(x) = \varphi_{ss_{k+1}^-}(x) + 1,$$

$$\frac{\partial\varphi_{ss_{k+1}}(x)}{\partial s_{k+1}} = f(s_{k+1},\varphi_{ss_{k+1}^-}(x)),$$

we obtain, for $j = k + 1$:

$$\frac{\partial\varphi_{st}(x)}{\partial s_{k+1}} = \partial_1\Phi(s_{k+1},t;\varphi_{ss_{k+1}}(x)) + \partial_3\Phi(s_{k+1},t;\varphi_{ss_{k+1}}(x))\,\frac{\partial\varphi_{ss_{k+1}}(x)}{\partial s_{k+1}}$$

$$= \exp\left\{\int_{s_{k+1}}^{t} \partial_2 f(r,\varphi_{sr}(x))\,\mathrm{d}r\right\}\Big[f(s_{k+1},\varphi_{ss_{k+1}^-}(x))$$

$$-f(s_{k+1},\varphi_{ss_{k+1}}(x))\Big]. \qquad \square$$

We will write $\varphi_t(x)$ for $\varphi_{0t}(x)$. It is well known that $\{\varphi_t(x),\ t \in [0,1]\}$ is a Markov process (see, e.g., Protter (1977), Section 5, for a proof in a more general situation). Carlen and Pardoux (1990) studied, using Malliavin calculus on the Poisson space, the law of the solution of Poisson-driven equations. In particular, they obtained that on the set $\{N_1 \geqslant 1\}$ the law of $\varphi_1(x)$ is absolutely continuous with respect to the Lebesgue measure, provided $\partial_2 f$ never vanishes. We generalise here this result proving that the law of $\varphi_t(x)$ is the weighted sum of a Dirac-δ and an absolutely continuous probability, without using the Malliavin calculus formalism. Moreover, we only need $f(t,x) \neq f(t,x+1)$, $\forall t, \forall x$, instead of $|\partial_2 f| > 0$.

Proposition 2.4. *Let f be a function satisfying hypotheses* (H_1') *and* (H_2'). *Assume moreover that $f(t,x) \neq f(t,x+1)$, $\forall t, \forall x$. Let $\varphi(x) = \{\varphi_t(x),\ t \in [0,1]\}$ be the solution to* (2.1) *for $s = 0$. Then, for all $t > 0$, the distribution function of $\varphi_t(x)$ can be written as*

$$F(y) = \mathrm{e}^{-t}F^{\mathrm{D}}(y) + (1 - \mathrm{e}^{-t})F^{\mathrm{C}}(y) \tag{2.4}$$

with

$$F^{\mathrm{D}}(y) := \mathbf{1}_{[\Phi(0,t;x),\infty)}(y)$$

and

$$F^{\mathrm{C}}(y) := (\mathrm{e}^{t} - 1)^{-1}\int_{-\infty}^{y}\sum_{n=1}^{\infty}\frac{t^n}{n!}h_n(r)\,\mathrm{d}r,$$

where h_n is the density function of the law of $\varphi_t(x)$ conditioned to $N_t = n$.

Proof. Let $S_1, S_2, \ldots$ be the jump times of $\{N_t,\ t \in [0,1]\}$. We know from Proposition 2.3 that on the set $\{N_t = n\}$ $(n = 1, 2, \ldots)$ we have $\varphi_t(x) = G(S_1, \ldots, S_n)$ for some continuously differentiable function G, and that

$$\partial_1 G(s_1, \ldots, s_n) = \exp\left\{\int_{s_1}^{t} \partial_2 f(r, \varphi_r(x))\,\mathrm{d}r\right\}\left[f(s_1, \varphi_{s_1^-}(x)) - f(s_1, \varphi_{s_1}(x))\right].$$

The hypothesis $f(t,x) \neq f(t,x+1)$, $\forall t, \forall x$ implies $|\partial_1 G| > 0$.

It is known that, conditionally to $\{N_t = n\}$, $(S_1, \ldots, S_n)$ follows the uniform distribution on $D_n = \{0 < s_1 < \cdots < s_n \leqslant t\}$. If we define $T(s_1, \ldots, s_n) = (z_1, \ldots, z_n)$, with $z_1 = G(s_1, \ldots, s_n)$ and $z_i = s_i$, $2 \leqslant i \leqslant n$, then $(Z_1, \ldots, Z_n) = T(S_1, \ldots, S_n)$ is a random vector with density

$$h(z_1, \ldots, z_n) = n! t^{-n} |\partial_1 s_1(z_1, \ldots, z_n)| \mathbf{1}_{T(D_n)}(z_1, \ldots, z_n)$$

and therefore $\varphi_t(x)$ is absolutely continuous on $\{N_t = n\}$, for every $n \geqslant 1$, with conditional density

$$h_n(z_1) = \mathbf{1}_{G(D_n)}(z_1) \int\int \cdots \int n! t^{-n} |\partial_1 s_1(z_1, \ldots, z_n)| \mathbf{1}_{T(D_n)}(z_1, \ldots, z_n)\,\mathrm{d}z_2 \ldots \mathrm{d}z_n.$$

Now,

$$\begin{aligned} F(y) &= \sum_{n=0}^{\infty} P\{\varphi_t(x) \leqslant y / N_t = n\} P\{N_t = n\} \\ &= \mathrm{e}^{-t} P\{\varphi_t(x) \leqslant y / N_t = 0\} + \mathrm{e}^{-t} \sum_{n=1}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{y} h_n(r)\,\mathrm{d}r \\ &= \mathrm{e}^{-t} \mathbf{1}_{[\Phi(0,t;x), \infty)}(y) + \mathrm{e}^{-t} \int_{-\infty}^{y} \sum_{n=1}^{\infty} \frac{t^n}{n!} h_n(r)\,\mathrm{d}r \end{aligned}$$

and (2.4) follows. □

Remark 2.5. If the hypothesis $f(t,x) \neq f(t,x+1)$, $\forall t$, $\forall x$, does not hold, then the conclusion of Proposition 2.4 is not necessarily true: Consider for instance the equation

$$\varphi_t = \int_0^t f(\varphi_r)\,\mathrm{d}r + N_t$$

with $f(n) = 0$ for $n = 0, 1, 2, \ldots$, whose solution is $\varphi \equiv N$. More generally, under hypotheses (H_1) and (H_2), the condition $f(t,x) = f(t,x+1)$, $\forall t$, $\forall x$ is sufficient for the process φ to have discrete laws, and in that case $\varphi_t(x) = \Phi(0,t;x) + N_t$.

3. The equation with boundary condition

In this section we establish first an easy existence and uniqueness theorem, based on Corollary 2.2 above, when the initial condition in our equation is replaced by a boundary condition. Then we prove in this situation the analogue of Propositions 2.3(3) and 2.4 on the differentiability with respect to the jump times and the absolute continuity of the laws (Propositions 3.2 and 3.3, respectively).

Theorem 3.1. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a measurable function satisfying hypotheses* (H_1) *and* (H_2) *of Section* 2 *with Lipschitz constant K. Let $\psi:\mathbb{R}\to\mathbb{R}$ be a continuous function such that*:

(H_3) *ψ satisfies one of the following one-sided Lipschitz or inverse-Lipschitz conditions*:

(a) $x>y \;\Rightarrow\; \psi(x)-\psi(y)\leqslant\eta\cdot(x-y)$, *for some real constant η such that* $\eta<\mathrm{e}^{-K}$.

(b) $x>y \;\Rightarrow\; \psi(x)-\psi(y)\geqslant\eta\cdot(x-y)$, *for some real constant η such that* $\eta>\mathrm{e}^{K}$.

Then there exists a unique càdlàg process $X=\{X_t,\ t\in[0,1]\}$ that solves the boundary value problem

$$X_t = X_0+\int_0^t f(s,X_s)\,\mathrm{d}s+N_t,\quad t\in[0,1],$$
$$X_0=\psi(X_1). \tag{3.1}$$

Proof. According to Corollary 2.2, for every $x\in\mathbb{R}$ there exists a unique càdlàg process $\varphi(x)=\{\varphi_t(x),\ t\in[0,1]\}$ satisfying

$$\varphi_t(x)=x+\int_0^t f(r,\varphi_r(x))\,\mathrm{d}r+N_t,\quad t\in[0,1].$$

From part (3) of that corollary and hypothesis (H_3), it is clear that the function $x\mapsto x-\psi(\varphi_1(\omega,x))$ has a unique fix point, that we define as $X_0(\omega)$. It follows that (3.1) has the unique solution $X_t(\omega)=\varphi_t(\omega,X_0(\omega))$. □

In the next two propositions, we assume the regularity hypotheses (H_1') and (H_2') on f. Moreover, we also require the boundary function ψ to be continuously differentiable.

Proposition 3.2. *Let f: $[0,1]\times\mathbb{R}\to\mathbb{R}$ and $\psi:\mathbb{R}\to\mathbb{R}$ satisfy hypotheses* (H_1'), (H_2') *of Section* 2 *and*

(H_3') *ψ is continuously differentiable with $\psi'\leqslant\eta<\mathrm{e}^{-K}$ or $\psi'\geqslant\eta>\mathrm{e}^{K}$, for some constant η, and let $X=\{X_t,\ t\in[0,1]\}$ be the solution to* (3.1). *Then,*

(1) *Let $n\in\{1,2,\ldots\}$. On the set $\{N_1=n\}$, X_0 can be regarded as a function $X_0(s_1,\ldots,s_n)$ defined on $\{0<s_1<\cdots<s_n\leqslant 1\}$, where $s_j=S_j(\omega)$ are the jump times of $N(\omega)$ in $[0,1]$. This function is continuously differentiable, and for any $j=1,2,\ldots,n$*:

$$\frac{\partial X_0}{\partial s_j}=\frac{\psi'(X_1)\exp\{\int_{s_j}^1\partial_2 f(r,X_r)\,\mathrm{d}r\}[f(s_j,X_{s_j^-})-f(s_j,X_{s_j})]}{1-\psi'(X_1)\exp\{\int_0^1\partial_2 f(r,X_r)\,\mathrm{d}r\}}. \tag{3.2}$$

(2) *Let $t\in(0,1]$ and $n,k\in\{0,1,\ldots\}$ such that $n+k\geqslant 1$. On the set $\{N_t=n\}\cap\{N_1-N_t=k\}$, X_t can be regarded as a function $X_t(s_1,\ldots,s_{n+k})$ defined on $\{0<s_1<\cdots<s_{n+k}\leqslant 1\}$, where $s_j=S_j(\omega)$ are the jump times of $N(\omega)$ in $[0,1]$.*

This function is continuously differentiable, and for any $j = 1, 2, \ldots, n+k$:

$$\frac{\partial X_t}{\partial s_j} = \exp\left\{\int_0^t \partial_2 f(r, X_r)\,\mathrm{d}r\right\}\frac{\partial X_0}{\partial s_j} + \exp\left\{\int_{s_j}^t \partial_2 f(r, X_r)\,\mathrm{d}r\right\}[f(s_j, X_{s_j^-}) - f(s_j, X_{s_j})]\mathbf{1}_{\{1\leqslant j\leqslant n\}}. \tag{3.3}$$

Proof. Since $X_0 = \psi(\varphi_1(X_0))$, we have

$$\frac{\partial X_0}{\partial s_j} = \frac{\psi'(\varphi_1(X_0))\partial\varphi_1(x)/\partial s_j|_{x=X_0}}{1 - \psi'(\varphi_1(X_0))\,\mathrm{d}\varphi_1(x)/\mathrm{d}x_{|_{x=X_0}}}$$

and (3.2) follows from Proposition 2.3, (2) and (3). On the other hand, from $X_t = \varphi_t(X_0)$,

$$\frac{\partial X_t}{\partial s_j} = \left.\frac{\mathrm{d}\varphi_t(x)}{\mathrm{d}x}\right|_{x=X_0}\frac{\partial X_0}{\partial s_j} + \left.\frac{\partial\varphi_t(x)}{\partial s_j}\right|_{x=X_0}\mathbf{1}_{\{1\leqslant j\leqslant n\}}$$

and we find (3.3) immediately. □

Proposition 3.3. *Let* $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ *and* $\psi:\mathbb{R}\to\mathbb{R}$ *satisfy hypotheses* (H_1'), (H_2') *and*

(H_3'') ψ *is of continuously differentiable and* $\psi' < 0$ *or* $0 < \psi' \leqslant \eta < \mathrm{e}^{-K}$ *or* $\psi' \geqslant \eta > \mathrm{e}^{K}$, *for some constant* η.

Assume that $f(t,x) \neq f(t, x+1)$, $\forall t$, $\forall x$. *Let* x^* *be the unique solution to* $x = \psi(\Phi(0,1;x))$. *Let* X *be the solution to* (3.1). *Then,*

(1) *The distribution function of* X_0 *is*

$$F_{X_0}(x) = \mathrm{e}^{-1}F_{X_0}^{\mathrm{D}}(x) + (1-\mathrm{e}^{-1})F_{X_0}^{\mathrm{C}}(x),$$

with

$$F_{X_0}^{\mathrm{D}}(x) = \mathbf{1}_{[x^*,\infty)}(x)$$

and

$$F_{X_0}^{\mathrm{C}}(x) = (\mathrm{e}-1)^{-1}\int_{-\infty}^{x}\sum_{n=1}^{\infty}\frac{h_n(r)}{n!}\,\mathrm{d}r,$$

where h_n *is the density of* X_0 *conditioned to* $N_1 = n$.

(2) *The distribution function of* X_t, $t\in(0,1]$ *is*

$$F_{X_t}(x) = \mathrm{e}^{-1}F_{X_t}^{\mathrm{D}}(x) + (1-\mathrm{e}^{-1})F_{X_t}^{\mathrm{C}}(x)$$

with

$$F_{X_t}^{\mathrm{D}}(x) = \mathbf{1}_{[\Phi(0,t;x^*),\infty)}(x)$$

and

$$F^{\mathrm{C}}_{X_t}(x) = \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-1}}\left[\mathrm{e}^{-(1-t)}\int_{-\infty}^{x}\sum_{n=1}^{\infty}\frac{(1-t)^n}{n!}h_{0n}(r)\,\mathrm{d}r + \int_{-\infty}^{x}\sum_{n=1}^{\infty}\frac{h_n(r)}{n!}\,\mathrm{d}r\right],$$

where h_{0n} is the density of X_t conditioned to $N_t=0$, $N_1=n$, and h_n is the density of X_t conditioned to $N_t=n$.

Proof. Under the given hypotheses, and from Proposition 3.2, we have $|\partial X_t/\partial s_1| > 0$, for every $t\in[0,1]$, on the sets $\{N_1=n\}$, $(n=1,2,\ldots)$. We deduce, as in Proposition 2.4, that the laws of X_t are absolutely continuous on these sets. On $\{N_1=0\}$ one has clearly $X_t \equiv \Phi(0,t;x^*)$.

The formula for F_{X_0} is obtained as in Proposition 2.4, but conditioning to $\{N_1 = n\}$. The formula for F_{X_t}, $t>0$, is also obtained in a straightforward manner as in Proposition 2.4, but starting with the decomposition

$$\begin{aligned} F_{X_t}(x) &= P\{X_t \leqslant x, N_1 = 0\} \\ &\quad + \sum_{n=1}^{\infty} P\{X_t\leqslant x/N_t=0, N_1-N_t=n\}\mathrm{e}^{-1}\frac{(1-t)^n}{n!} \\ &\quad + \sum_{n=1}^{\infty} P\{X_t\leqslant x/N_t=n\}\mathrm{e}^{-t}\frac{t^n}{n!}. \qquad \square \end{aligned}$$

4. The reciprocal property

Let us recall the definition of a reciprocal process. We introduce first the usual notation for conditional independence: Let $(\Omega,\mathfrak{F},P)$ be a probability space and let $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{B}$ be sub-σ-fields of $\mathfrak{F}$ such that

$$P\{A_1\cap A_2/\mathfrak{B}\} = P\{A_1/\mathfrak{B}\}P\{A_2/\mathfrak{B}\}$$

for any $A_1\in\mathfrak{F}_1$, $A_2\in\mathfrak{F}_2$. Then we say that the σ-fields $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are conditionally independent given $\mathfrak{B}$, and we write

$$\mathfrak{F}_1 \underset{\mathfrak{B}}{\perp\!\!\!\perp} \mathfrak{F}_2.$$

Definition 4.1. A stochastic process $\{X_t,\ t\in[0,1]\}$ is called *reciprocal* if for every $0\leqslant a<b\leqslant 1$,

$$\sigma\{X_t,\ t\in[a,b]\} \underset{\sigma\{X_a,X_b\}}{\perp\!\!\!\perp} \sigma\{X_t,\ t\in(a,b)^{\mathrm{c}}\}.$$

Any Markov process is reciprocal. This fact was stated in Jamison (1970) for continuous processes. We have not been able to find in the literature a proof for the general case. We give here a short one.

Proposition 4.2. *Any Markov process is reciprocal. The converse is not true.*

Proof. An easy example of a reciprocal process which is not Markov is $\{N_t + N_1,\ t \in [0,1]\}$. Suppose now that X is a Markov process in $[0,1]$. Fix $0 \leqslant a < b \leqslant 1$ and set $\mathfrak{F}_1 := \sigma\{X_t,\ t \in [0,a]\}$, $\mathfrak{F}_2 := \sigma\{X_t,\ t \in [a,b]\}$, $\mathfrak{F}_3 := \sigma\{X_t,\ t \in [b,1]\}$, $\mathfrak{B} := \sigma\{X_a, X_b\}$. Using elementary properties of the conditional independence (see, for instance, Rozanov, 1982) one finds that the Markov property implies

$$\mathfrak{F}_1 \underset{\mathfrak{B}}{\perp\!\!\!\perp} \mathfrak{F}_2 \vee \mathfrak{F}_3, \quad \mathfrak{F}_2 \underset{\mathfrak{B}}{\perp\!\!\!\perp} \mathfrak{F}_3 \quad \text{and} \quad \mathfrak{F}_1 \underset{\mathfrak{B}}{\perp\!\!\!\perp} \mathfrak{F}_3. \tag{4.1}$$

Let us show that (4.1) implies that $\mathfrak{F}_1 \vee \mathfrak{F}_3 \underset{\mathfrak{B}}{\perp\!\!\!\perp} \mathfrak{F}_2$, and the proof will be complete. Indeed, $\mathfrak{F}_1 \vee \mathfrak{F}_3$ is generated by the π-system of sets of the form $A_1 \cap A_3$, with $A_1 \in \mathfrak{F}_1$ and $A_3 \in \mathfrak{F}_3$. Taking $A_1 \in \mathfrak{F}_1$, $A_2 \in \mathfrak{F}_2$, and $A_3 \in \mathfrak{F}_3$, we obtain

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3/\mathfrak{B}) &= P(A_1/\mathfrak{B})P(A_2 \cap A_3/\mathfrak{B}) \\ &= P(A_1/\mathfrak{B})P(A_2/\mathfrak{B})P(A_3/\mathfrak{B}) \\ &= P(A_1 \cap A_3/\mathfrak{B})P(A_2/\mathfrak{B}). \qquad \square \end{aligned}$$

For stochastic boundary value problems driven by the Wiener process, two different methods have been used to determine in which cases the solution is a reciprocal process. The first is based in the anticipating version of the Girsanov theorem (see Kusuoka, 1982). The other one makes use of the characterisation of conditional independence given in Lemma 4.3 below. Both methods can be applied to the Wiener analogue of our equation (see Nualart and Pardoux (1991) and Alabert et al. (1995), respectively), and they allow to prove, assuming f satisfies (H_1'), (H_2'), and ψ is continuously differentiable and non-increasing, that the solution is a reciprocal process if and only if $f(t,\cdot)$ is an affine function for every t or $\psi' \equiv 0$.

In the Poisson case we will see that there is no such a neat characterisation of the reciprocal property. Affine functions f always lead to reciprocal processes, but 1-periodic and other nonlinear functions f also do; however, introducing additional hypotheses on a nonlinear drift, we find a class of equations for which one can ensure that the reciprocal property does not hold.

Our plan is as follows: in Lemmas 4.3–4.6, we introduce some tools that we will needed. The first three lemmas are not new, and we refer the reader to the original papers for the proofs. Then we prove that if the drift is affine or 1-periodic in the second variable, then the solution is reciprocal (Theorems 4.7 and 4.8); we exhibit other drifts coefficients with the same property (Example 4.9), and finally we state a negative result under some conditions on the drift (Theorem 4.10).

The following result was proved in Alabert et al. (1995).

Lemma 4.3. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two independent sub-σ-fields in a probability space $(\Omega, \mathfrak{F}, P)$. Consider two functions $g_1 : \mathbb{R} \times \Omega \to \mathbb{R}$ and $g_2 : \mathbb{R} \times \Omega \to \mathbb{R}$ such that g_i is $\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{F}_i$-measurable, $i = 1,2$, and they satisfy the following conditions for some* $\varepsilon_0 > 0$:

(C_1) *For every $y_1, y_2 \in \mathbb{R}$, the random variables $g_1(y_1,\cdot)$ and $g_2(y_2,\cdot)$ possess absolutely continuous distributions and their corresponding densities $f_1(y_1,z)$ and $f_2(y_2,z)$ are locally bounded in $\mathbb{R}^2$.*

(C$_2$) *For any* $|\xi| < \varepsilon_0$, $|\theta| < \varepsilon_0$, *the system*

$$y_2 - g_1(y_1, \omega) = \xi,$$
$$y_1 - g_2(y_2, \omega) = \theta$$

has a unique solution $(y_1, y_2) \in \mathbb{R}^2$, *for almost all* $\omega \in \Omega$.

(C$_3$) *For almost all* $\omega \in \Omega$, *the functions* $y_1 \mapsto g_1(y_1, \omega)$ *and* $y_2 \mapsto g_2(y_2, \omega)$ *are continuously differentiable, and there exists an integrable random variable* H *such that*

$$\sup_{\substack{|y_1 - g_2(y_2,\omega)| < \varepsilon_0 \\ |y_2 - g_1(y_1,\omega)| < \varepsilon_0}} |1 - g_1'(y_1, \omega) g_2'(y_2, \omega)|^{-1} \leqslant H(\omega) \quad a.s.$$

Let Y_1 *and* Y_2 *be the random variables determined, according with* (C$_2$), *by the system*

$$Y_2(\omega) = g_1(Y_1(\omega), \omega),$$
$$Y_1(\omega) = g_2(Y_2(\omega), \omega).$$

Then, the following statements are equivalent:

(i) $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *are conditionally independent given* Y_1 *and* Y_2.

(ii) *There exist two functions* $F_1 : \mathbb{R}^2 \times \Omega \to \mathbb{R}$, $F_2 : \mathbb{R}^2 \times \Omega \to \mathbb{R}$, *which are* $\mathfrak{B}(\mathbb{R}^2) \otimes \mathfrak{F}_i$*-measurable*, $i = 1, 2$, *such that*

$$|1 - g_1'(Y_1) g_2'(Y_2)| = F_1(Y_1, Y_2, \omega) F_2(Y_1, Y_2, \omega) \quad a.s.$$

If, in addition, $1 - g_1'(Y_1) g_2'(Y_2)$ *has constant sign, then* (i) *and* (ii) *are equivalent to*

(iii) *At least one of the random variables* $g_1'(Y_1)$, $g_2'(Y_2)$ *is a.s. constant, with respect to the conditional law given* Y_1 *and* Y_2.

Lemma 4.3 can be generalised to n-dimensional variables (see Ferrante and Nualart, 1997). In case the distributions of Y_1 and Y_2 are discrete, the following Lemma (Alabert and Nualart, 1992) states that the conditional independence always holds true. This result is valid with any measurable space in place of $\mathbb{R}$.

Lemma 4.4. *Let* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *be two independent sub-σ-fields in a probability space* $(\Omega, \mathfrak{F}, P)$. *Consider two functions* $g_1 : \mathbb{R} \times \Omega \to \mathbb{R}$ *and* $g_2 : \mathbb{R} \times \Omega \to \mathbb{R}$ *such that* g_i is $\mathfrak{B}(\mathbb{R}) \otimes \mathfrak{F}_i$*-measurable*, $i = 1, 2$, *and that the system*

$$y_2 - g_1(y_1, \omega) = 0,$$
$$y_1 - g_2(y_2, \omega) = 0$$

has a unique solution $(y_1, y_2) \in \mathbb{R}^2$, *for almost all* $\omega \in \Omega$. *Let* Y_1 *and* Y_2 *be the random variables determined by this system, and assume they have discrete laws. Then* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *are conditionally independent given* Y_1 *and* Y_2.

We will also make use of the following tool, first employed in a context similar to ours by Ferrante and Nualart (1997) to prove that certain specific two-dimensional linear equation with boundary condition and Wiener noise does not have a reciprocal solution, thus solving a problem left open in Ocone and Pardoux (1989).

Lemma 4.5. *Let $\mathfrak{F}_1$, $\mathfrak{F}_2$ and $\mathfrak{G}$ three sub-σ-fields in a probability space $(\Omega, \mathfrak{F}, P)$. Let $B = B_1 \cap B_2$, where $B_1 \in \mathfrak{F}_1$ and $B_2 \in \mathfrak{F}_2$ and $P(B) > 0$. Denote by $\mathfrak{F}_{i|_B} := \{A \cap B : A \in \mathfrak{F}_i\}$, $i = 1, 2$. If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are conditionally independent given $\mathfrak{G}$ in $(\Omega, \mathfrak{F}, P)$, then $\mathfrak{F}_{1|_B}$ and $\mathfrak{F}_{2|_B}$ are conditionally independent given $\mathfrak{G}_{|_B}$ in $(B, \mathfrak{F}_{|_B}, P(\cdot/B))$.*

It should be noted that one cannot deduce the global conditional independence from the local conditional independence for all B_i in a partition.

Lemma 4.6. *If $\xi = \{\xi_t,\ t \in [0,1]\}$ has independent increments and g is a Borel function, then $X := \{g(\xi_1) + \xi_t,\ t \in [0,1]\}$ is a reciprocal process.*

Proof. Fix $0 \leqslant a < b \leqslant 1$. Set $\mathfrak{B} := \sigma\{X_a, X_b\}$, $\mathfrak{F}^i_{ab} := \sigma\{\xi_t - \xi_a,\ t \in [a,b]\}$, and $\mathfrak{F}^e_{ab} := \sigma\{\xi_t,\ t \in [0,a];\ \xi_t - \xi_b,\ t \in [b,1]\}$. Since $\mathfrak{F}^i_{ab} \perp\!\!\!\perp \mathfrak{F}^e_{ab}$ and $\sigma\{\xi_b - \xi_a\} \subset \mathfrak{F}^i_{ab}$, we have

$$\mathfrak{F}^i_{ab} \underset{\sigma\{\xi_b - \xi_a\}}{\perp\!\!\!\perp} \mathfrak{F}^e_{ab} \vee \sigma\{\xi_b - \xi_a\}. \tag{4.2}$$

Similarly, (4.2) and $\sigma\{\xi_b - \xi_a\} \subset \mathfrak{B} \subset \mathfrak{F}^e_{ab} \vee \sigma\{\xi_b - \xi_a\}$ imply

$$\mathfrak{F}^i_{ab} \vee \mathfrak{B} \underset{\mathfrak{B}}{\perp\!\!\!\perp} \mathfrak{F}^e_{ab} \vee \mathfrak{B}.$$

We use finally that $\sigma\{X_t,\ t \in [a,b]\} \subset \mathfrak{F}^i_{ab} \vee \mathfrak{B}$ and $\sigma\{X_t,\ t \in (a,b)^c\} \subset \mathfrak{F}^e_{ab} \vee B$, and the proof is complete. $\square$

Theorems 4.7, 4.8 and 4.10 are the main results of this paper.

Theorem 4.7. *Assume f has the form $f(t,x) = \alpha(t) + \beta(t)x$, where α and β are continuous functions, and that Eq. (3.1) has a unique solution X. Then, X is a reciprocal process.*

Proof. If f has the given form, Eq. (3.1) has a unique solution if and only if for almost every $\omega \in \Omega$, there exists a unique value $X_0(\omega)$ verifying $X_0(\omega) = \psi(A(1)(X_0(\omega) + \xi_1(\omega)))$, where

$$A(t) := \exp\left\{\int_0^t \beta(r)\,\mathrm{d}r\right\}, \quad \xi_t := \int_0^t A(r)^{-1}[\alpha(r)\,\mathrm{d}r + \mathrm{d}N_r]$$

and in that case the solution is given by $X_t = A(t)[X_0 + \xi_t]$, a.s.

Therefore, there exists a Borel function g such that $X_0 = g(\xi_1)$, a.s., and we can write

$$X_t = A(t)[g(\xi_1) + \xi_t].$$

Since ξ has independent increments, we conclude from Lemma 4.6 that X is reciprocal. $\square$

Theorem 4.8. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ verify hypotheses* (H$_1$), (H$_2$) *and (H$_3$), and that $f(t,x) = f(t, x+1)$, $\forall t$, $\forall x$. Let $X = \{X_t,\ t \in [0,1]\}$ be the solution to* (3.1). *Then,*

(1) *For every $t \in [0,1]$, X_t has a discrete law.*

(2) *X is a reciprocal process.*

Proof. In this case, the solution to (3.1) satisfies $X_1 = \Phi(0,1;\psi(X_1)) + N_1$ (see Remark 2.5). Together with (H_3), this implies that X_1 is N_1-measurable, and we can write $X_t = \Phi(0,t;\chi(N_1)) + N_t = \Phi(0,t;\chi(N_1) + N_t)$ for some Borel function χ. Claim (1) follows immediately. Claim (2) follows from (1), applying Lemma 4.4, and it is also a consequence of Lemma 4.6, since $\Phi(0,t,\cdot)$ is invertible. □

Example 4.9. The process $X = \{\alpha t + N_t - \frac{1}{2}N_1,\ t \in [0,1]\}$, where $\alpha \in [0,1/2)$, is reciprocal, because it is the solution to (3.1) with $f(t,x) \equiv \alpha$ and $\psi(x) = \alpha - x$, and we can apply either Theorem 4.7 or 4.8. But X also solves

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_r)\,\mathrm{d}r + N_t, \quad t \in [0,1],\\ X_0 &= \alpha - X_1 \end{aligned} \tag{4.3}$$

for any function $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(x) \equiv \alpha$ for every $x \in \bigcup_{k\in\mathbb{Z}} [k/2, k/2+\alpha]$. In particular, $N_t - \frac{1}{2}N_1$ solves (4.3) for any function f vanishing on the set $\{k/2,\ k \in \mathbb{Z}\}$.

This example shows that we cannot expect to find a characterisation of the reciprocal property similar to that of equations driven by white noise. Intuitively, the reason is that there is not a one-to-one correspondence between drift and boundary coefficients on one side and solution processes on the other, unlike in the additive Wiener case.

The next theorem states a negative result for the reciprocal property. We restrict ourselves to autonomous drift coefficients.

Theorem 4.10. *Let $f:\mathbb{R}\to\mathbb{R}$ and $\psi:\mathbb{R}\to\mathbb{R}$ be measurable functions satisfying*:
(1) *f is continuously differentiable, $f > 0$ and $|f'| \leqslant K$, for some constant K.*
(2) *$f(x) \neq f(x+1)$, $\forall x$.*
(3) *ψ is continuously differentiable with $\psi' < 0$ or $0 < \psi' \leqslant \eta < \mathrm{e}^{-K}$ or $\psi' \geqslant \eta > \mathrm{e}^{K}$, for some η.*
(4) *The function $F:\mathbb{R}\to\mathbb{R}$ defined by*

$$F(x) = \frac{f'(x+1)f(x) - f(x+1)f'(x)}{f(x+1) - f(x)},$$

satisfies $F(x) \neq F(x+1+\varepsilon)$ for all $\varepsilon > 0$ and $x \in \mathbb{R}$.
Then, the solution to

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_r)\,\mathrm{d}r + N_t, \quad t \in [0,1],\\ X_0 &= \psi(X_1) \end{aligned} \tag{4.4}$$

is not a reciprocal process.

Proof. We will split the proof into several steps. Let $X = \{X_t,\ t \in [0,1]\}$ be the solution to (4.4), which exists and is unique under the given hypotheses, by virtue of Theorem 3.1.

Step 1: Fix $0 \leqslant a < b \leqslant 1$, and denote

$$\mathfrak{F}^i_{ab} := \sigma\{N_t - N_a,\ a \leqslant t \leqslant b\}, \quad \mathfrak{F}^e_{ab} := \sigma\{N_t,\ 0 \leqslant t \leqslant a; N_t - N_b,\ b \leqslant t \leqslant 1\}.$$

Then,

$$\sigma\{X_t,\ t\in[a,b]\} \underset{\sigma\{X_a,X_b\}}{\perp\!\!\!\perp} \sigma\{X_t,\ t\in(a,b)^{\rm c}\} \quad \text{if and only if} \quad \mathfrak{F}^i_{ab} \underset{\sigma\{X_a,X_b\}}{\perp\!\!\!\perp} \mathfrak{F}^e_{ab}.$$

Proof of Step 1. The ‘only if’ part is obvious, because

$$\mathfrak{F}^i_{ab}\subseteq\sigma\{X_t,\ t\in[a,b]\} \quad \text{and} \quad \mathfrak{F}^e_{ab}\subseteq\sigma\{X_t,\ t\in(a,b)^{\rm c}\}.$$

Conversely, the ‘if’ part follows from the relations

$$\sigma\{X_t,\ t\in[a,b]\}\subseteq\mathfrak{F}^i_{ab}\vee\sigma\{X_a,X_b\},$$

$$\sigma\{X_t,\ t\in(a,b)^{\rm c}\}\subseteq\mathfrak{F}^e_{ab}\vee\sigma\{X_a,X_b\}$$

and elementary properties of the conditional independence. □

Step 2: Set $B=B_1\cap B_2$, where $B_1:=\{N_b-N_a=2\}\in\mathfrak{F}^i_{ab}$ and $B_2:=\{N_a=2\}\cap\{N_1-N_b=0\}\in\mathfrak{F}^e_{ab}$. Then,

$$\mathfrak{F}^i_{ab} \underset{\sigma\{X_a,X_b\}}{\perp\!\!\!\perp} \mathfrak{F}^e_{ab} \Rightarrow \mathfrak{F}^i_{ab|B} \underset{\sigma\{X_a,X_b\}_{|B}}{\perp\!\!\!\perp} \mathfrak{F}^e_{ab|B}.$$

Proof of Step 2. This is a consequence of Lemma 4.5. □

Step 3: On the probability space $(B,\mathfrak{F}_{|B},P(\cdot/B))$, the σ-fields $\mathfrak{F}^i_{ab_{|B}}$ and $\mathfrak{F}^e_{ab_{|B}}$ are independent and the functions

$$(y_1,\omega)\mapsto g_1(y_1,\omega):=\varphi_{ab}(\omega,y_1),$$

$$(y_2,\omega)\mapsto g_2(y_2,\omega):=\varphi_a(\omega,\psi(\varphi_{b1}(\omega,y_2))),$$

verify conditions (C_1) – (C_3) of Lemma 4.3.

Proof of Step 3. From Lemma 4.5, the given σ-fields are independent, since $\mathfrak{F}^i_{ab}$ and $\mathfrak{F}^e_{ab}$ are independent in $(\Omega,\mathfrak{F},P)$. Let us check properties (C_1) – (C_3).

(C_3): From Corollary 2.3(2), we have that for any ω, the mappings g_1 and g_2 are continuously differentiable and

$$g_1'(y_1,\omega)=\exp\left\{\int_a^b f'(\varphi_{ar}(\omega,y_1))\,\mathrm{d}r\right\},$$

$$g_2'(y_2,\omega)=\exp\left\{\int_0^a f'(\varphi_r(\omega,\psi(\varphi_{b1}(\omega,y_2))))\,\mathrm{d}r+\int_b^1 f'(\varphi_{br}(\omega,y_2))\,\mathrm{d}r\right\}$$
$$\times\psi'(\varphi_{b1}(\omega,y_2)).$$

When $\psi'\geqslant\eta>\mathrm{e}^K$, we get

$$1-g_1'(y_1,\omega)g_2'(y_2,\omega)\leqslant1-\eta\mathrm{e}^{-K}<0$$

and when $\psi'<0$ or $0<\psi'\leqslant\eta<\mathrm{e}^{-K}$, we get

$$1-g_1'(y_1,\omega)g_2'(y_2,\omega)\geqslant1-\eta\mathrm{e}^{K}>0,$$

so that $|1-g_1'(y_1,\omega)g_2'(y_2,\omega)|^{-1}$ is bounded. Notice also that $1-g_1'(y_1,\omega)g_2'(y_2,\omega)$ has always constant sign.

(C_2): It is enough to show that the function $J(y) := y - g_2(\xi + g_1(y,\omega),\omega) + \theta$ has a unique zero. We have just seen that in all cases $|J'| > \varepsilon$, for some $\varepsilon > 0$. Therefore, J vanishes at exactly one point.

(C_1): The random variable $g_1(y,\cdot)$ restricted to $(B, \mathfrak{F}_{|B}, P(\cdot/B))$, satisfies, for $i=3,4$,

$$\frac{\partial g_1(y)}{\partial s_i} = \exp\left\{\int_{s_i}^{b} f'(\varphi_{ar}(y))\,\mathrm{d}r\right\}\left[f(\varphi_{as_i^-}(y)) - f(\varphi_{as_i}(y))\right],$$

from which we deduce, following the arguments of Proposition 2.4, that it is absolutely continuous and its density $f_1(y,z)$ is locally bounded in $\mathbb{R}^2$. Analogously, for the random variable $g_2(y,\cdot)$ restricted to $(B, \mathfrak{F}_{|B}, P(\cdot/B))$, one has, for $i = 1,2$,

$$\frac{\partial g_2(y)}{\partial s_i} = \exp\left\{\int_{s_i}^{a} f'(\varphi_r(\psi(\varphi_{b1}(y))))\,\mathrm{d}r\right\}$$
$$\times[f(\varphi_{as_i^-}(\psi(\varphi_{b1}(y)))) - f(\varphi_{as_i}(\psi(\varphi_{b1}(y))))]$$

and $g_2(y,\cdot)$ must be absolutely continuous with density $f_2(y,z)$ locally bounded in $\mathbb{R}^2$. □

Step 4: The solution X to (4.4) *is not a reciprocal process.*

Proof of Step 4. Assume X is a reciprocal process, and fix $0 < a < b < 1$. By Steps 1 and 2, this would imply

$$\mathfrak{F}^{i}_{ab_{|B}} \underset{\sigma\{X_a,X_b\}_{|B}}{\perp\!\!\!\perp} \mathfrak{F}^{e}_{ab_{|B}},$$

where $B = \{N_a = 2\} \cap \{N_b - N_a = 2\} \cap \{N_1 - N_b = 0\}$. For each $\omega \in B$, the path $N(\omega)$ jumps exactly twice, at some times s_1, s_2, in $[0,a]$, exactly twice, at times s_3, s_4, in $(a,b]$, and never in $(b,1]$.

By Step 3, we can apply Lemma 4.3 in the given situation with $Y_1 = X_a$ and $Y_2 = X_b$ to deduce that at least one of the random variables

$$g_1'(X_a) = \exp\left\{\int_a^b f'(X_r)\,\mathrm{d}r\right\}$$

or

$$g_2'(X_b) = \exp\left\{\int_0^a f'(X_r)\,\mathrm{d}r\right\}\exp\left\{\int_b^1 f'(X_r)\,\mathrm{d}r\right\}\psi'(X_1)$$

is a.s. constant, with respect to the conditional law given X_a and X_b.

Assume $g_2'(X_b)$ is constant. Notice that

$$\exp\left\{\int_b^1 f'(X_r)\,\mathrm{d}r\right\}\psi'(X_1) = \exp\left\{\int_b^1 f'(X_r)\,\mathrm{d}r\right\}\psi'(\varphi_{b1}(X_b))$$

depends only on X_b, since $N_1 - N_b = 0$. Therefore, $\int_0^a f'(X_r)\,\mathrm{d}r$ must be constant, given X_a and X_b. For $t \in [0,a]$ we can write

$$X_t = \psi(\varphi_{b1}(X_b)) + \int_0^t f(X_r)\,\mathrm{d}r + N_t.$$

Applying the change of variables formula (Corollary 2.2(4)) to the functions $G(x) := \int_0^x \mathrm{d}r/f(r)$ and $H(x) := \log f(x)$ one obtains

$$G(X_a) = G(\psi(\varphi_{b1}(X_b))) + a + [G(X_{s_1}) - G(X_{s_1^-})] + [G(X_{s_2}) - G(X_{s_2^-})]$$

and

$$H(X_a) = H(\psi(\varphi_{b1}(X_b))) + \int_0^a f'(X_r)\,\mathrm{d}r + [H(X_{s_1}) - H(X_{s_1^-})] \\ +[H(X_{s_2}) - H(X_{s_2^-})].$$

Differentiating with respect to $X_{s_1^-}$, with X_a and X_b given, we find

$$[G'(X_{s_1}) - G'(X_{s_1^-})] + [G'(X_{s_2}) - G'(X_{s_2^-})]\,\frac{\mathrm{d}X_{s_2^-}}{\mathrm{d}X_{s_1^-}} = 0$$

and

$$[H'(X_{s_1}) - H'(X_{s_1^-})] + [H'(X_{s_2}) - H'(X_{s_2^-})]\,\frac{\mathrm{d}X_{s_2^-}}{\mathrm{d}X_{s_1^-}} = 0,$$

from which we obtain

$$\frac{H'(X_{s_1}) - H'(X_{s_1^-})}{G'(X_{s_1}) - G'(X_{s_1^-})} = \frac{H'(X_{s_2}) - H'(X_{s_2^-})}{G'(X_{s_2}) - G'(X_{s_2^-})}$$

or, equivalently,

$$F(X_{s_1^-}) = \frac{f'(X_{s_1^-}+1)f(X_{s_1^-}) - f(X_{s_1^-}+1)f'(X_{s_1^-})}{f(X_{s_1^-}+1) - f(X_{s_1^-})} \\ = \frac{f'(X_{s_2^-}+1)f(X_{s_2^-}) - f(X_{s_2^-}+1)f'(X_{s_2^-})}{f(X_{s_2^-}+1) - f(X_{s_2^-})} = F(X_{s_2^-}). \tag{4.5}$$

But, from hypothesis (1), for every $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is strictly increasing. Therefore $X_{s_2^-} > X_{s_1} = X_{s_1^-} + 1$, and equality (4.5) contradicts hypothesis (4). Hence $g_2'(X_b)$ cannot be constant, a.s., with respect to the conditional law given X_a and X_b.

For $t \in [a,b]$, we can write

$$X_t = X_a + \int_a^t f(X_r)\,\mathrm{d}r + (N_t - N_a)$$

and

$$G(X_b) = G(X_a) + (b-a) + [G(X_{s_3}) - G(X_{s_3^-})] + [G(X_{s_4}) - G(X_{s_4^-})],$$

$$H(X_b) = H(X_a) + \int_a^b f'(X_r)\,\mathrm{d}r + [H(X_{s_3}) - H(X_{s_3^-})] + [H(X_{s_4}) - H(_{s_4^-})].$$

A similar reasoning as above shows that $g_1'(X_a)$ cannot be constant, a.s., given X_a y X_b. We conclude that X is not a reciprocal process. $\square$

Remark 4.11. Hypothesis (4) in Theorem 4.10 holds, for instance, if F is one-to-one. Consider, on the other hand, the functions $f,\psi:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=e^x\mathbf{1}_{(-\infty,0)}(x)+(1+x)\mathbf{1}_{[0,\infty)}(x) \quad \text{and} \quad \psi(x)=cx,$$

with $0<c<1/e$, which satisfy all conditions of Theorem 4.10, except (4). Notice that, for every $\omega\in\Omega$,

$$\varphi_1(\omega,0)=\int_0^1 f(\varphi_r(\omega,0))\,\mathrm{d}r+N_1\geqslant 0$$

and that the function $h(x):=x-\psi(\varphi_1(\omega,x))$ is strictly increasing (this can be seen using Proposition 2.3(2)). Since $h(0)\leqslant 0$, the unique zero of h is positive and we find that $X_0(\omega)\geqslant 0$. This implies that the whole path $X_t(\omega)$ is non-negative and X solves

$$X_t=X_0+\int_0^t (1+X_r)\,\mathrm{d}r+N_t, \quad t\in[0,1],$$
$$X_0=\psi(X_1),$$

as well. We know, according to Theorem 4.7, that the solution is a reciprocal process.

Remark 4.12. All the results of Sections 2–4 can be easily generalised to the case when N_t is a Poisson process with jumps of arbitrary fixed size.

5. Poisson noise with finite source

Assume that the random disturbance of our equations has only a finite source of Poisson impulses. That means, the driving noise is a Poisson process N conditioned to $N_1\leqslant k$, for some fixed $k\in\mathbb{N}$. This change does not affect any of the results on existence, uniqueness and regularity of Sections 2 and 3. Propositions 2.4 and 3.3 on the law of the solution also hold true with the obvious changes.

Concerning the Markov and reciprocal properties, the situation is more complex in this case. On the one hand, the path space of the noise is poorer, and conditional independence properties are more likely to hold. On the other, such a noise, though Markovian, does not possess independent increments. The application of the main tools of Section 4 (Lemmas 4.3, 4.4 and 4.6) rely heavily in this feature of the driving process, and therefore we cannot use them here. Lemma 4.5 still can be applied in the following way:

If the solution X to (3.1) is reciprocal and the event $B=\{N_1\leqslant k\}$ is X_1-measurable, then X is still reciprocal when conditioning to B. Indeed, Lemma 4.5 applied to the set B leads to this conclusion. For instance, this is the case in the situation of Theorem 4.8 (f is 1-periodic), where one can write $N_1=X_1-\Phi(0,1;\psi(X_1))$.

In the remaining of this section, we will discuss the case $k=1$. That means, we consider the following set-up. Let $\{\xi_t,\ t\in[0,1]\}$ be a process defined in some probability space $(\Omega,\mathfrak{F},P)$, all of whose paths are $\{0,1\}$-valued, non-decreasing, and càdlàg, with $\xi_0\equiv 0$, and whose law is that of a standard Poisson process N conditioned to $N_1\leqslant 1$. In particular, $P\{\xi_t=1\}=t/2$ and $P\{\xi_t=0\}=1-t/2$. It is easily checked that ξ is a Markov process but does not have independent increments. We can define on Ω a random variable S such that $S=t$ if ξ jumps at time t, and takes a special value v if ξ does not have jumps. Then, $S=v$ with probability 1/2 and, conditioned to $S \neq v$, S is uniformly distributed on $[0,1]$.

We consider the problem

$$X_t = X_0 + \int_0^t f(r, X_r)\,\mathrm{d}r + \xi_t, \quad t \in [0,1],$$

$$X_0 = \psi(X_1). \tag{5.1}$$

Under hypotheses (H_1) – (H_3) of Theorem 3.1, there exists obviously a unique solution, which can be expressed as a function of S by

$$X_t(S) = \begin{cases} \Phi(0,t;x_0), & \text{if } S = v, \\ \Phi(0,t;X_0(S)), & \text{if } t < S \leqslant 1, \\ \Phi(S,t;\Phi(0,S;X_0(S))+1), & \text{if } 0 < S \leqslant t, \end{cases} \tag{5.2}$$

where x_0 is the unique solution to $x=\psi(\Phi(0,1;x))$, $X_0(S)$ is the unique solution to $x=\psi(\Phi(S,1;\Phi(0,S;x)+1))$, and Φ is defined by Eq. (2.2).

We are going to show first that X is always a reciprocal process (Theorem 5.1); then we will see that it is in fact a Markov process in a variety of situations (Theorem 5.2), but that this is not always the case (Example 5.3).

Theorem 5.1. *Under hypotheses* (H_1) – (H_3) *defined in Sections* 2 *and* 3, *the solution* X *to* (5.1) *is a reciprocal process. Moreover, if* ψ *is constant,* X *is a Markov process.*

Proof. Fix $0 \leqslant a < b \leqslant 1$. Set $\mathfrak{F}^i = \sigma\{X_t,\ t \in [a,b]\}$ and $\mathfrak{F}^e = \sigma\{X_t,\ t \in (a,b)^c\}$. Denote $A := \{S \in (a,b]^c\}$. Let us show first that for every $F_i \in \mathfrak{F}^i$, $F_i \cap A$ is (X_a, X_b)-measurable, and that for every $F_e \in \mathfrak{F}^e$, $F_e \cap A^c$ is (X_a, X_b)-measurable.

The set A coincides with $B := \{X_b = \Phi(a,b;X_a)\}$. Indeed, it is clear that $A \subset B$; for the other inclusion, notice that on A^c, one has $X_b = \Phi(S,b;\Phi(a,S;X_a)+1)$. Since Φ is one-to-one in the third variable (Proposition 2.1, 3), $A^c \subset B^c$. We conclude from $A=B$ that the sets A and A^c belong to the σ-field $\sigma\{X_a, X_b\}$, and that for any $t \in [a,b]$, we can write $X_t = \Phi(a,t;X_a)\mathbf{1}_A + X_t\mathbf{1}_{A^c}$. Therefore, for each $C \in \mathfrak{F}\mathfrak{B}(\mathbb{R})$,

$$\{X_t \in C\} \cap A = \{\Phi(a,t;X_a) \in C\} \in \sigma\{X_a, X_b\}.$$

This yields $F_i \cap A \in \sigma\{X_a, X_b\}$, for any $F_i \in \mathfrak{F}^i$. Similarly, we obtain $F_e \cap A \in \sigma\{X_a, X_b\}$, for any $F_e \in \mathfrak{F}^e$.

Now we can prove that $\mathfrak{F}^i$ and $\mathfrak{F}^e$ are conditionally independent given (X_a, X_b): Let $F_i \in \mathfrak{F}^i$ and $F_e \in \mathfrak{F}^e$. We have,

$$\begin{aligned} P(F_i \cap F_e / X_a, X_b) &= P(F_i \cap F_e \cap A / X_a, X_b) + P(F_i \cap F_e \cap A^c / X_a, X_b) \\ &= \mathbf{1}_{F_i \cap A} P(F_e / X_a, X_b) + \mathbf{1}_{F_e \cap A^c} P(F_i / X_a, X_b) \\ &= \mathbf{1}_A P(F_i / X_a, X_b) P(F_e / X_a, X_b) + \mathbf{1}_{A^c} P(F_e / X_a, X_b) P(F_i / X_a, X_b) \\ &= P(F_e / X_a, X_b) P(F_i / X_a, X_b). \end{aligned}$$

Finally, from the reciprocal property applied to $a=0$ and to $0<b\leqslant 1$, we conclude that X is a Markov process when ψ is a constant. □

Theorem 5.2. *Assume f satisfies hypotheses* (H_1') – (H_3') *of Sections* 2 *and* 3. *Then, the solution to Eq.* (5.1) *is a Markov process in each of the following cases*:
(a) $|\partial_2 f|>0$ *and* $\psi'<0$.
(b) $\partial_2 f>0$ *and* $0<\psi'\leqslant\eta<\mathrm{e}^{-K}$, *for some* η.
(c) $\partial_2 f<0$ *and* $\psi'\geqslant\eta>\mathrm{e}^{K}$, *for some* η.
(d) $\partial_2 f<0$ *and* $0<\psi'\leqslant\eta<(K\mathrm{e}^K+1)^{-1}$, *for some* η.
(e) $\partial_2 f>0$ *and* $\psi'\geqslant\eta>(1-K)^{-1}>1$, *for some* η.

Proof. We continue using representation (5.2) of X_t as a function of the jump time S. We will show in all cases that $\sigma\{X_t\}=\sigma\{S\}$ for all $t\in[0,1]$; in other words, S is determined by each X_t, so that the σ-fields generated by X_t are the same for every t, and the Markov property is trivial.

Notice first that, using the injectivity of ψ and $\Phi(0,t;\cdot)$, one finds easily that for all $t,s\in[0,1]$, $X_t(v)\neq X_t(s)$. From formulae (3.2) and (3.3), the variables X_0 and X_1, as functions of s, are invertible, so that we get $\sigma\{X_0\}=\sigma\{X_1\}=\sigma\{S\}$. From (3.3), we see that, for $t\in(0,1)$, X_t, as a function of s, is strictly monotone in each of the intervals $(0,t]$ and $(t,1]$. We will show that

$$\sup_{s\in(t,1]} X_t(s)<\inf_{s\in(0,t]} X_t(s) \tag{5.3}$$

and the proof will be complete. Using the monotonicity of X_t, (5.3) can be written in terms of Φ (using (5.2)) as

$$\Phi(0,t;X_0(t))\vee\Phi(0,t;X_0(1))<\Phi(0,t;X_0(0^+)+1)\wedge\Phi(0,t;X_0(t))+1. \tag{5.4}$$

In cases (a) with $\partial_2 f<0$, (b) and (c), X_0 is decreasing, and (5.4) holds because $x\mapsto\Phi(0,t;x)$ is strictly increasing, by Proposition 2.1(3).

In case (a) with $\partial_2 f>0$, $s\mapsto X_t(s)$ is decreasing in $[0,t]$ and increasing in $(t,1]$, so that proving (5.4) reduces to prove $\Phi(0,t;X_0(1))<\Phi(0,t;X_0(t))+1$. Suppose the converse inequality; then,

$$X_{1^-}(1)=\Phi(t,1;\Phi(0,t;X_0(1)))\geqslant\Phi(t,1;\Phi(0,t;X_0(t))+1)=X_1(t),$$

which implies $X_1(1)>X_1(t)$. Since ψ is decreasing, we get $X_0(1)<X_0(t)$, a contradiction, since X_0 is increasing in the present case.

In case (d), $s\mapsto X_t(s)$ is increasing in both intervals $[0,t]$ and $(t,1]$, so that (5.4) amounts to $\Phi(0,t;X_0(1))<\Phi(0,t;X_0(0^+)+1)$. From (3.2), we have $X_0(1)-X_0(0^+)\leqslant\eta(1-\eta)^{-1}K\mathrm{e}^K<1$, and the inequality holds true. Case (e) can be done similarly. □

Theorem 5.2 remains true if we replace conditions $\partial_2 f>0$ and $\partial_2 f<0$ by $f(t,x+1)-f(t,x)>0$, $\forall t,x$, and $f(t,x+1)-f(t,x)<0$, $\forall t,x$, respectively.

The following example shows that the solution to (5.1) is not always a Markov process.

Example 5.3. Consider the problem

$$X_t = X_0 - \int_0^t X_r\,\mathrm{d}r + \xi_t, \quad t \in [0,1]$$

$$X_0 = (e - 1/2)X_1.$$

The solution is given by

$$X_t(S) = \begin{cases} 0, & \text{if } S = v, \\ 2\mathrm{e}^{1-t}\mathrm{e}^S, & \text{if } 0 < S \leqslant t, \\ (2e-1)\mathrm{e}^{-t}\mathrm{e}^S & \text{if } t < S \leqslant 1. \end{cases}$$

It is clear that $\sigma\{X_0\} = \sigma\{X_1\} = \sigma\{S\}$, hence $\sigma\{X_r,\ r \leqslant \frac{1}{2}\} = \sigma\{X_r,\ r \geqslant \frac{1}{2}\} = \sigma\{S\}$. However, one can easily see that $\sigma\{X_{1/2}\}$ is strictly included in $\sigma\{S\}$, and this implies that X cannot be a Markov process.

Remark 5.4. Theorem 5.1 shows that the solution to (3.1), which is not reciprocal in general, can enjoy this property when the noise is conditioned to a finite number of jumps. The same happens with the Markov property: The process $X_t = N_t - \frac{1}{2}N_1$, considered in Example 4.9 is not Markov, but it becomes Markov when conditioning to $N_1 \leqslant 1$.

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