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Exit Times from Equilateral Triangles*

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Abstract. In this paper we obtain a closed form expression of the expected exit time of a Brownian motion from equilateral triangles. We consider first the analogous problem for a symmetric random walk on the triangular lattice and show that it is equivalent to the ruin problem of an appropriate three player game. A suitable scaling of this random walk allows us to exhibit explicitly the relation between the respective exit times. This gives us the solution of the related Poisson equation.

Key Words. Exit time, Random walk, Brownian motion, Poisson equation.

AMS Classification. Primary 60J65, Secondary 35C05.

1. Introduction

In this paper we obtain the expected exit time of a random walk and a Brownian motion from an equilateral triangle. The random walk we consider is not on the regular integer lattice, but on the triangular lattice, where the random walk takes a step in each of the possible six directions with equal probability.

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As is well known, the exit time problem for a random walk on the one-dimensional line, with steps of size 1 taken in unit time, can also be stated as a ruin problem, namely, Peter and Paul play a game with capitals \$$a$ and \$$b$ respectively and according to the outcome of a toss of a coin, a dollar changes hands—with the game being played until one of the two is bankrupt. If it takes unit time to toss a coin, then the distribution of the "time to ruin" is the same as the exit time from the interval $]-a, b[$ of a random walk with steps of unit length starting at the origin.

We first generalize the ruin problem to three players. Let Peter, Paul and Mary play the following game. First a pair is chosen from the three players, with each pair being equally probable of being chosen. According to the outcome of a toss of a coin, a dollar changes hands. Then a pair is chosen again from the three and a coin tossed to determine who amongst the pair wins and who loses a dollar. This game of alternately choosing a pair and tossing a coin is continued until one of the three is bankrupt. If a, b and c are the respective capitals of Peter, Paul and Mary, we are interested in determining the "time to ruin", assuming each toss takes a unit time.

In Section 2 we show that the above problem is equivalent to obtaining the exit time from an equilateral triangle of a random walk problem on the "triangular lattice" in the plane. For this ruin problem (or the equivalent random walk), the discrete harmonic equations yielding as a solution the expected time of bankruptcy (or the expected exit time) may be written quite easily (see (4) below) and we know the solution to this set of equations.

In Section 3 we look at the exit time of a two-dimensional standard Brownian motion from an equilateral triangle. We use that an appropriate time and scale change of the random walk on the triangular lattice approximates in law a two-dimensional Brownian motion, as happens in the one-dimensional case. Then we show that the laws of the exit times also converge (Proposition 3) and that so do their expectations (Proposition 5). We obtain the following result:

Theorem 1. *The expectation of the exit time from an equilateral triangle of the standard Brownian motion B_t on the plane is given by*

$$2\sqrt{3}\lambda_1\lambda_2\lambda_3 A, \tag{1}$$

where A is the area of the triangle and $(\lambda_1, \lambda_2, \lambda_3)$ are the barycentric coordinates of the starting point with respect to the vertices of the triangle.

It is well known that the expected exit time of a Brownian motion from the triangle, as a function of the starting point, is the unique solution to the Poisson equation

$$\tfrac{1}{2}\Delta u = -1 \tag{2}$$

that vanishes on the boundary of the triangle. Therefore, expression (1) provides such a solution, which, to our knowledge, has never been found before. Of course, once this formula is conjectured, the proof of Theorem 1 is immediate by checking that it satisfies (2) and the boundary condition. However, our main goal is to exhibit the relation between the exit times of planar Brownian motion and triangular random walks and how (1) is derived from the latter, with the help of its relation with the gambling problem described above.

In the three-dimensional analogous problem (exit time of a Brownian motion from a tetrahedron), we have not been able to obtain an explicit result as in Theorem 1 above, although we can, in principle, compute the solution of the corresponding harmonic equations (analogous to (4)) for tetrahedrons with side lengths of specified integers. One might think that $k\lambda_1\lambda_2\lambda_3\lambda_4$, where k is some constant, solves the three-dimensional Poisson equation, but this is readily seen to be false.

The Dirichlet problem in a triangle for the equation $\Delta f + \lambda f = 0$ was studied by Pinsky [6] and it can also be related to the exit time from triangles [3] and to the motion of Brownian particles in a circle with annihilation when they collide [1].

2. The Lattice and the Harmonic Equations

Let S be a positive integer and construct a triangle $\triangle_S$, each of whose sides is of length S. Consider the regular triangular lattice with edges of unit length. We place the triangle $\triangle_S$ on the lattice such that each of the vertices of $\triangle_S$ is a vertex of the lattice (see Figure 1).

We label the edges 1, 2 and 3. A vertex of the triangular lattice in $\triangle_S$ is given the coordinate (a, b, c) where a (respectively b and c) is the length of a shortest path comprising of edges of the triangular lattice from the vertex to the edge labelled 1 (respectively 2 and 3). A little thought shows that if (a, b, c) is the label of a vertex in the triangle $\triangle_S$, then $a+b+c = S$. Note that this is just a scaled barycentric coordinate system with respect to the three vertices of the triangle. Clearly, if (a, b, c) and (a', b', c') are two neighbouring vertices of the triangular lattice in $\triangle_S$, then $|a-a'|+|b-b'|+|c-c'| = 2$. We perform a random walk on this lattice. Starting from a vertex (a, b, c) we take a step to one of the neighbouring six vertices with probability $1/6$ each, steps being taken independent of one another.

This random walk problem is indeed equivalent to the "ruin" problem of Peter, Paul and Mary. To see this, let the fortunes of Peter, Paul and Mary be respectively a, b and c. After a game the fortunes change to (a', b', c') with probability $1/6$ where $|a - a'| + |b - b'| + |c - c'| = 2$.

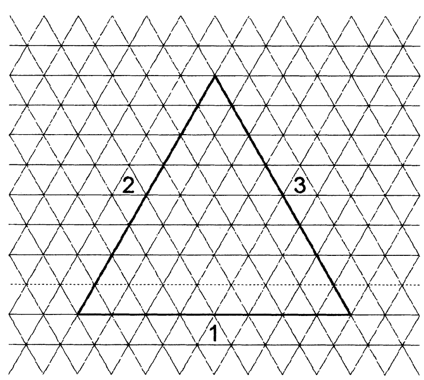

Figure 1. The triangular lattice.

The notion of "ruin", i.e. one of Peter, Paul or Mary becoming bankrupt translates to the random walk setting as the walk hitting the boundary of the triangle $\triangle_S$.

In the cartesian (x, y)-plane, if the triangle $\triangle_S$ is such that one vertex coincides with the origin, another vertex is at the point $(S, 0)$ and the third vertex is at the point $(S/2, \sqrt{3}S/2)$, then the vertex (a, b, c) of the previous construction corresponds to the cartesian coordinates $(b + a/2, \sqrt{3}a/2)$. Conversely, if (α, β) are the cartesian coordinates of a vertex of the triangular lattice, then the scaled barycentric coordinates are $((2/\sqrt{3})\beta, \alpha - (1/\sqrt{3})\beta, S - \alpha - (1/\sqrt{3})\beta)$.

Now let $h(a, b, c)$ be the expected time to ruin of the Peter, Paul and Mary problem when their respective fortunes to begin with are \$$a$, \$$b$ and \$$c$. Clearly we have

$$h(a, b, c) = 0 \qquad \text{whenever} \quad \min\{a, b, c\} = 0. \tag{3}$$

Moreover, for $a, b, c > 0$, an argument based on conditioning immediately yields

$$\begin{aligned} h(a, b, c) = 1 + \tfrac{1}{6}\{&h(a-1, b+1, c) + h(a+1, b-1, c) + h(a, b-1, c+1) \\ &+ h(a, b+1, c-1) + h(a-1, b, c+1) \\ &+ h(a+1, b, c-1)\}. \end{aligned} \tag{4}$$

It may be easily seen that

$$h(a, b, c) = \frac{3abc}{a+b+c} \tag{5}$$

is the unique bounded solution to the above equation (4) with boundary condition (3). The uniqueness of the solution follows from the fact that the difference between two distinct bounded solutions must be a bounded function whose value at any point equals the average of its values at the neighbouring points; and hence, from (3), it is identically zero.

Equation (4) above may be thought of as the discrete analogue of the Poisson equation. Indeed, let $\mathbf{P}$ be the averaging operator and let $\mathbf{I}$ be the identity operator, then

$$\begin{aligned} \mathbf{P}h(a, b, c) = \tfrac{1}{6}\{&h(a-1, b+1, c) + h(a+1, b-1, c) \\ &+ h(a, b-1, c+1) + h(a, b+1, c-1) \\ &+ h(a-1, b, c+1) + h(a+1, b, c-1)\}. \end{aligned} \tag{6}$$

The operator $\mathbf{A} := \mathbf{P} - \mathbf{I}$ may be taken to be the discrete analogue of $\frac{1}{2}\Delta$ where Δ is the Laplace operator (see, e.g. [2].

In terms of the above notation, (4) reduces to $\mathbf{A}h = -1$; thus the analogous equation in the continuous case is

$$\tfrac{1}{2}\Delta u = -1 \tag{7}$$

and the boundary condition (3) takes the form

$$u(x) = 0 \qquad \text{for } x \text{ on the boundary of } \triangle_S\,. \tag{8}$$

For this Poisson problem, it is well known that the probabilistic representation of its *unique* solution $u(x)$ is the expected exit time from any general triangle $\triangle_S$ of a Brownian motion starting at x (see, e.g., [2]). Thus our theorem gives a closed form solution of this problem. We have not been able to find in the literature a solution of the above Poisson problem by purely analytic means (see Chapter III in [8] and Chapter 19 in [9] for a discussion of analytic methods in the study of Poisson equations).

3. Convergence of the Expected Exit Times

Now let S be a positive real number. With the help of the gambling model of Section 2, we are going to find an explicit expression for the expected exit time from the equilateral triangle $\triangle_S$ of a planar Brownian motion, thereby obtaining an explicit solution of (7) with boundary condition (8).

To this end, we first define a sequence of approximating random walks converging in law to a Brownian motion. Then we prove that their expected exit times must converge to the corresponding value for the limiting process. The sequence of random walks is chosen so that the expected exit times are approximated using the gambling model. Therefore, the limit will give us the exit time for a Brownian motion.

For a process X on the plane, denote by $T(X, \triangle_S)$ its exit time from $\triangle_S$:

$$T(X, \triangle_S) := \inf\{t\colon\ X_t \notin \triangle_S\}.$$

Let $\{\xi_n\}_{n \in N}$ be a sequence of independent, identically distributed random vectors taking values $(\cos(k\pi/3), \sin(k\pi/3))$ for $k = 1, \dots, 6$ with equal probability. Note that ξ_n may be thought of as a single step of the random walk on the triangular lattice.

Fix $(\alpha, \beta) \in \mathbb{R}^2$ and, for each $n \in \mathbb{N}$, let $\{Y_t^n,\ t \geq 0\}$ be the processes defined as follows:

$$Y_t^n = (\alpha, \beta) + \sqrt{2/n}\left(\sum_{i=1}^{\lfloor t \rfloor} \xi_i + (t - \lfloor t \rfloor)\xi_{\lfloor t \rfloor + 1}\right), \tag{9}$$

where $\lfloor\cdot\rfloor$ denotes the integer part.

For the process

$$X_t^n := Y_{nt}^n, \qquad t \geq 0,$$

we have

Proposition 2. *The sequence of processes $\{X_t^n,\ t \geq 0\}$ converges in law to a standard Brownian motion starting at (α, β).*

Proof. First, notice that though the two scalar components of ξ_i are not independent, they are uncorrelated. This is enough to prove that the sequence X_n converges in law to the standard Brownian motion on $\mathbb{R}^2$. Indeed, the convergence of the finite-dimensional distributions follows from the multidimensional central limit theorem by standard arguments. On the other hand, tightness of the sequence X_n follows from the same property for its scalar components, which is a consequence of their convergence in law given by Donsker's Invariance Principle for $\mathbb{R}$-valued random walks (see, e.g. pp. 70–71 of [4]). □

In order to prove the convergence of the expected exit times, we show first that $T(X^n, \triangle_S)$ converges in law to $T(B, \triangle_S)$ where B is the standard Brownian motion starting from (α, β). Then we prove that $\{T(X^n, \triangle_S)\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence of random variables. This gives us the convergence of $\{E[T(X^n, \triangle_S)]\}_{n\in\mathbb{N}}$ to $E[T(B, \triangle_S)]$.

The next proposition establishes the convergence of the exit times. Kushner and Dupuis [5, p. 260] present an argument based on the law of iterated logarithms of a one-dimensional Brownian motion to exhibit the convergence of stopping times of an approximating Markov chain. Although their argument may be adapted in our two-dimensional setting, we use another direct argument.

Proposition 3. *The following convergence in law holds*:

$$T(X^n, \triangle_S) \Longrightarrow T(B, \triangle_S).$$

Proof. In this proof we adopt, for convenience, the following notation: $\triangle$ denotes the open triangle of edge length S, $\partial\triangle$ its boundary and $\bar{\triangle}$ its closure.

Let $T^n := T(X^n, \triangle)$ and $T := T(B, \triangle)$ denote respectively the exit times from $\triangle$ of the process X^n_t and the Brownian motion B defined above.

To get the stated weak convergence, it suffices to prove

$$P\{T^n > t\} \longrightarrow P\{T > t\} \qquad \text{for all} \quad t > 0. \tag{10}$$

Let P^n be the law of X^n and let P_B be the law of B, both on the set $\mathcal{C} := C([0,\infty), \mathbb{R}^2)$ of continuous functions on $[0,\infty)$ with values in the plane, equipped with the Borel σ-algebra. In terms of P^n and P_B, we have

$$P\{T^n > t\} = P\{X^n_s \in \triangle \text{ for all } s \le t\} = P^n(A),$$

where $A = \{x \in \mathcal{C}\colon\ x(0) = (\alpha, \beta) \text{ and } x(s) \in \triangle \text{ for all } s \le t\}$. Analogously, $P\{T > t\} = P_B(A)$.

The set A may also be expressed as

$$A = \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \{x \in \mathcal{C}\colon\ x(0) = (\alpha, \beta) \text{ and } \\ x(s) \in \triangle_{-1/k},\ \text{for all } s \le t, s \in D_m\}, \tag{11}$$

where D_m is the set of dyadic numbers of order m and $\triangle_{-1/k} := \triangle - \bar{V}_{1/k}(\partial\triangle)$, $V_{1/k}(\partial\triangle)$ being the $1/k$-neighbourhood of $\partial\triangle$. The right-hand side of (11) is obviously Borel measurable because the set in curly braces is in fact a cylindrical set.

By Proposition 2 we have the weak convergence of $\{P^n\}_n$ to P_B, which is equivalent to

$$P^n(E) \longrightarrow P_B(E) \qquad \text{for all Borel sets } E \text{ with} \quad P_B(\partial E) = 0.$$

Hence, in order to show (10) it suffices to prove that A is a continuity set with respect to P_B, i.e. $P_B(\partial A) = 0$. To this end observe first that $\mathcal{C}$ is a complete separable metric space under the metric d, defined by

$$d(x, y) := \sum_{k=1}^{\infty} \frac{1}{2^k} (\|y - x\|_k \wedge 1),$$

where $\|\ \|_s$ denotes the supremum norm on the compact interval $[0, s]$.

Now we establish that A is an open set. Indeed, fix x in A and let k_0 be such that $x(s) \in \triangle_{-1/k_0}$ for all $s \le t$ dyadic. Set $k = \lfloor t \rfloor + 1$ and take $\varepsilon = 1/2^{k+1}k_0$. If $d(x, y) < \varepsilon$, then

$$\|y - x\|_t \le \|y - x\|_k < 2^k \varepsilon = \frac{1}{2k_0},$$

i.e. $y(s) \in \triangle_{-1/2k_0}$, for all $s \le t$ dyadic, and therefore $y \in A$.

The closure of A is

$$\bar{A} = \{x \in \mathcal{C} : \ x(0) = (\alpha, \beta) \text{ and } x(s) \in \bar{\triangle} \text{ for all } s \le t\}$$

and its boundary is

$$\begin{aligned}\partial A &= \bar{A} - A \\ &= \{x \in \mathcal{C}\colon \ x(0) = (\alpha, \beta), x(s) \in \bar{\triangle} \text{ for all } s \le t \\ &\qquad \text{and } x(s') \in \partial \triangle \ \text{ for some } s' \le t\}.\end{aligned}$$

Decomposing $\partial\triangle$ as the disjoint union of its three edge segments r_1, r_2 and r_3, we obtain

$$\begin{aligned}P_B(\partial A) \le \sum_{i=1}^{3} P_B\{x \in \mathcal{C}\colon \ x(0) = (\alpha, \beta), x(s) \in \bar{\triangle} \text{ for all } s \le t \\ \text{and } x(s') \in r_i \text{ for some } s' \le t\}.\end{aligned}$$

Moreover, if H_{r_i} denotes the closed half-plane determined by r_i and containing the triangle $\bar{\triangle}_S$, then the following inclusion holds:

$$\begin{aligned}&\{x \in \mathcal{C}\colon \ x(0) = (\alpha, \beta), x(s) \in \bar{\triangle} \text{ for all } s \le t \text{ and } x(s') \in r_i \text{ for some } s' \le t\} \\ &\quad \subseteq \{x \in \mathcal{C}\colon \ x(0) = (\alpha, \beta), x(s) \in H_{r_i} \text{ for all } s \le t \\ &\qquad \text{and } x(s') \in r_i \text{ for some } s' \le t\}.\end{aligned}$$

Hence, it suffices to show that the sets of the form

$$\begin{aligned}&\{x \in \mathcal{C} : x(0) = (\alpha, \beta) \in H_r - r, \ \ x(s) \in H_r, \ \text{ for all } s \le t \\ &\ \text{and } x(s') \in r \text{ for some } s' \le t\},\end{aligned}$$

where H_r is the half-plane determined by the line r and the starting point, have measure zero under P_B.

By the rotational invariance property of Brownian motion, we can assume that r has the form $y = m$ for some positive constant m and that the Brownian motion starts at the origin, i.e. $x(0) = (0, 0)$. If we denote $x(s) =: (x^1(s), x^2(s))$ the two components of the planar process, then we want to show that

$$\begin{aligned}&P_B\{x \in \mathcal{C} : x(0) = (0, 0), \ x^2(s) \le m, \ \text{ for all } s \le t \\ &\qquad \text{and } x^2(s') = m \text{ for some } s' \le t\}\end{aligned}$$

vanishes. However, this is the probability that the maximum of a one-dimensional Brownian motion in the interval $[0, t]$ takes the value m, which is zero, because the law of this maximum is absolutely continuous. This proves the proposition. □

Lemma 4. *The sequence $\{T(X^n, \triangle_S)\}_{n\in\mathbb{N}}$ is uniformly integrable.*

Proof. Using the same notations we introduced in the proof of Proposition 3, let $T^n := T(X^n, \triangle_S)$ and $T := T(B, \triangle_S)$ denote respectively the exit times from $\triangle_S$ of the process X_t^n and the Brownian motion B, starting at (α, β).

It is well known that for the Wiener process $\forall\varepsilon > 0, \exists\delta > 0$ such that

$$P\{T < \varepsilon\} > \delta,$$

for all starting points (α, β),

From this and using Proposition 3, it follows that $\forall\varepsilon > 0, \exists\delta > 0$ such that for large enough n,

$$P\{T^n < \varepsilon\} > \delta,$$

for all starting points (α, β).

Taking $\varepsilon = 1$ and the corresponding δ, and applying iteratively the Markov property for the time homogeneous Markov process X^n, we have that, for each integer k,

$$P\{T^n > k\} = P\{T^n > 1\}^k \leq (1-\delta)^k.$$

Therefore, for each integer M,

$$\int_{\{T^n > M\}} T^n \, dP \leq \sum_{k=M}^{\infty} (k+1) P\{T^n > k\} \leq \sum_{k=M}^{\infty} (k+1)(1-\delta)^k,$$

which converges to zero as M tends to ∞. □

As discussed earlier, this yields

Proposition 5. $\lim_{n\to\infty} E[T(X^n, \triangle_S)] = E[T(B, \triangle_S)]$.

To find the limit of the sequence of expected exit times, in the following lemma we rewrite it in terms of a scaled version Z^n of Y^n given by

$$Z^n := \sqrt{n/2}\,Y^n. \tag{12}$$

In this scaled version the steps will be of unit size. This will allow us to use the properties relative to the random walk or the ruin problem described earlier.

Lemma 6. *The following equalities hold*:

$$T(X^n, \triangle_S) = \frac{1}{n} T(Y^n, \triangle_S) = \frac{1}{n} T(Z^n, \triangle_{\sqrt{n/2}S}). \tag{13}$$

Proof. We have from the definition of Z^n (see (12) and (9)),

$$Z^n_{T(Z^n,\triangle_{\sqrt{n/2}S})} = \sqrt{n/2}(\alpha, \beta) + \sum_{i=1}^{\lfloor T(Z^n,\triangle_{\sqrt{n/2}S})\rfloor} \xi_i + \lambda \xi_{\lfloor T(Z^n,\triangle_{\sqrt{n/2}S})\rfloor+1}$$

for some $\lambda \in [0, 1[$. Therefore, for the process $\tilde{Y}^n$ defined as Y^n but starting at $\sqrt{n/2}(\alpha, \beta)$, we have

$$\tilde{Y}^n_{T(Z^n,\triangle_{\sqrt{n/2}S})} = \sqrt{n/2}(\alpha, \beta) + \sqrt{2/n}\left(\sum_{i=1}^{\lfloor T(Z^n,\triangle_{\sqrt{n/2}S})\rfloor} \xi_i + \lambda \xi_{\lfloor T(Z^n,\triangle_{\sqrt{n/2}S})\rfloor+1}\right),$$

which is a point in the segment that joins Z^n_0 and $Z^n_{T(Z^n,\triangle_{\sqrt{n/2}S})}$, whose distance to Z^n_0 is $\sqrt{2/n}$ times the length of that segment. Hence, $\tilde{Y}^n_{T(Z^n,\triangle_{\sqrt{n/2}S})}$ lies on the boundary of the equilateral triangle of edge length S which is the translation of $\triangle_S$ by the vector $(1-\sqrt{2/n})\sqrt{n/2}(\alpha, \beta)$ (see Figure 2). Moreover, it is clear that $\tilde{Y}^n_t$ is in the interior of the above triangle for all $t < T(Z^n, \triangle_{\sqrt{n/2}S})$. Therefore, by a translation of this triangle back to the origin, we obtain

$$T(Y^n, \triangle_S) = T(Z^n, \triangle_{\sqrt{n/2}S}).$$

Now, by the definition of X^n, it is clear that

$$T(X^n, \triangle_S) = \frac{1}{n} T(Y^n, \triangle_S),$$

and the equalities (13) are proved. □

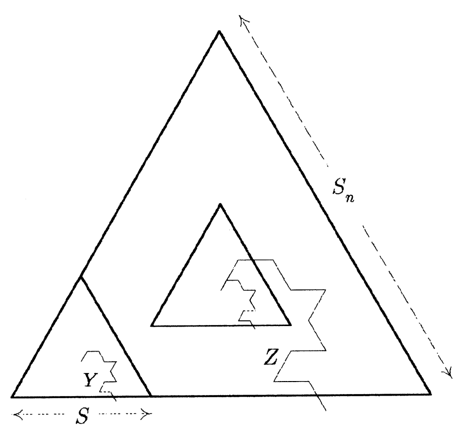

Figure 2. The random walks Z and Y. Here $S_n = \sqrt{n/2}S$.

In the two-dimensional case we are treating, we know the solution to the harmonic equations (4). This allows us to compute explicitly the expected exit times from triangles of integer size of the random walks performed by unit steps and starting from a vertex of the triangular lattice. Then we deduce from them the value of $E[T(B, \triangle_S)]$.

Proposition 7. *The limit of the expected exit times from $\triangle_S$ of the approximating random walks X^n is*

$$\frac{\sqrt{3}\beta(\alpha - (1/\sqrt{3})\beta)(S - \alpha - (1/\sqrt{3})\beta)}{S}, \tag{14}$$

where (α, β) is the starting point.

Proof. Recall that $T(Z^n, \triangle_{\sqrt{n/2}S})$ is the exit time from the triangle $\triangle_{\sqrt{n/2}S}$ of a random walk with unit steps starting at $\sqrt{n/2}(\alpha, \beta)$.

Clearly, the set

$$\{(\alpha, \beta) \in \triangle_S\colon \sqrt{n/2}(\alpha, \beta) \quad \text{is a vertex of the triangular lattice} \\ \text{for infinitely many } n\text{'s}\}$$

is dense in $\triangle_S$. The continuity of the expected exit time with respect to the starting point allows us to restrict ourselves to (α, β) belonging to the above dense set.

Fix n and consider the exit time $T(Z^n, \triangle_{\sqrt{n/2}S})$ starting at the vertex $\sqrt{n/2}(\alpha, \beta)$ of the lattice. To view this as a ruin problem we have to consider triangles with edges of integer length. So we take $m = \lfloor \sqrt{n/2}S \rfloor + 1$ and look at the exit times $T(Z^n, \triangle_{m-1})$ and $T(Z^n, \triangle_m)$. These can be interpreted as the times to ruin of the three player game explained in Section 2, with initial fortunes given by, for $k = m - 1, m$,

$$a_k = \sqrt{n/2}\frac{2}{\sqrt{3}}\beta,$$
$$b_k = \sqrt{n/2}\left(\alpha - \frac{1}{\sqrt{3}}\beta\right),$$
$$c_k = k - \sqrt{n/2}\left(\alpha + \frac{1}{\sqrt{3}}\beta\right).$$

Using formula (5) we find immediately that, for $k = m - 1, m$,

$$E[T(Z^n, \triangle_k)] = n\sqrt{3}\beta\left(\alpha - \frac{1}{\sqrt{3}}\beta\right)\frac{k - \sqrt{(n/2)}\alpha - \sqrt{(n/2)}(1/\sqrt{3})\beta}{k}.$$

On the other hand, applying Lemma 6,

$$\frac{1}{n}T(Z^n, \triangle_k) = T(X^n, \triangle_{k/\sqrt{n/2}}).$$

Then, using the inclusion $\triangle_{(m-1)/\sqrt{n/2}} \subseteq \triangle_S \subseteq \triangle_{m/\sqrt{n/2}}$ and taking limits, we obtain

$$\lim_{n\to\infty} E[T(X^n, \triangle_S)] = \frac{\sqrt{3}\beta(\alpha - (1/\sqrt{3})\beta)(S - \alpha - (1/\sqrt{3})\beta)}{S}. \qquad \square$$

It can easily be checked that $u(\alpha, \beta)$ given by the expression (14) solves the Poisson problem (7) with boundary condition (8). Expressing (14) in barycentric coordinates, and using Proposition 5, we immediately obtain Theorem 1.

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References

1. Balding, D. (1988), Diffusion-reaction in one dimension, J. Appl. Probab. 25:733–743.
2. Dynkin, E.B., and Yushkevich, A.A. (1969), Markov Processes: Theorems and Problems, Plenum, New York.
3. Hobson, D.G., and Werner, W. (1996), Non-colliding Brownian motions on the circle, Bull. London Math. Soc. 28:643–650.
4. Karatzas, I., and Shreve S.E. (1991), Brownian Motion and Stochastic Calculus, second edn., Springer-Verlag, New York.
5. Kushner, H.K., and Dupuis, P. (2001), Numerical Methods for Stochastic Control in Continuous Time, second edn., Springer-Verlag, New York.
6. Pinsky, M.A. (1980), The eigenvalues of an equilateral triangle, SIAM J. Math. Anal. 11:819–827.
7. Spitzer, F. (1976), Principles of Random Walk, second ed., Springer-Verlag, New York.
8. Treves, F. (1975), Basic Linear Partial Differential Equations, Academic Press, New York.
9. Triebel, H. (1986), Analysis and Mathematical Physics, Reidel, Leipzig.

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