STOCHASTIC DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS AND THE CHANGE OF MEASURE METHOD

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1. Introduction

The definition of several types of stochastic integrals for anticipating integrands put the basis for the development, in recent years, of an anticipating stochastic calculus. It is natural to consider, as an application, some problems that can be stated formally as stochastic differential equations, but that cannot have a sense within the theory of non-anticipating stochastic integrals. For example, this is the case if we impose to an s.d.e. an initial condition which is not independent of the driving process, or if we prescribe boundary conditions for the solution.

In this paper, we will try to survey the work already done concerning s.d.e. with boundary conditions, and to explain in some detail a method based in transformations and change of measure in Wiener space. An alternative approach is sketched in the last Section.

Transformations on Wiener space provide a natural method, among others, to prove existence and uniqueness results for nonlinear equations. At the same time, a Girsanov type theorem for not necessarily adapted transformations allows to study properties of the laws of the solutions from properties of the solution to an associated linear equation. A natural first question about these laws is to decide if they satisfy some kind of Markov (or conditional independence) property.

In Section 2, we introduce stochastic differential equations with boundary conditions, the particular instances that have been studied, and the kind of results obtained concerning conditional independence properties of the solutions. The short Section 3 outlines the idea of the method of transformations and change of measure. In Section 4, we recall the necessary elements of Wiener space analysis in order to enounce the Girsanov type theorem we want to apply. In Section 5, we describe the use of suitable transformations to obtain existence and uniqueness results for nonlinear equations. Section 6 is devoted to explain how the change of measure induced by the same transformations allows to derive characterizations of conditional independence properties. Finally, in Section 7 we describe briefly other alternatives to this method and the situations to which they have been applied.

Consider the following general problem:

$$(2.1)\qquad \left.\begin{array}{c} dX_t = f(t, X_t)\,dt + \sum_{i=1}^{k} \sigma_i(t, X_t) \circ dW_t^i \quad , \quad 0 \le t \le 1 \\ h(X_0, X_1) = 0 \end{array}\right\}$$

where $f, \sigma_i \colon [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $h \colon \mathbb{R}^{2d} \longrightarrow \mathbb{R}^d$, and $\{W_t,\ 0 \le t \le 1\}$ is a k-dimensional Wiener process ($k \le d$). The customary initial condition for X_0 is replaced by the boundary condition $h(X_0, X_1) = 0$.

By a *solution* to (2.1) we mean a d-dimensional continuous process X_t verifying the system

$$(2.2)\qquad \left.\begin{array}{c} X_t = \int_0^t f(s, X_s)\,ds + \sum_{i=1}^{k} \int_0^t \sigma_i(s, X_s) \circ dW_s^i \quad , \quad 0 \le t \le 1 \\ h(X_0, X_1) = 0 \end{array}\right\}$$

But here, unlike the initial-value problem, we cannot expect in general the existence of a solution adapted to the Wiener process, since the boundary condition makes X_0 depend on X_1, which in turn will depend on the whole Wiener process, through the integral equation. Therefore, the stochastic integral in (2.2) has to be understood as an anticipating stochastic integral. With the circle we denote, as usual, the Stratonovich anticipating integral.

As stated in the Introduction, two problems have been tackled concerning such equations: First, of course, the problem of existence and uniqueness of a solution; and secondly, to find sufficient and necessary conditions on the coefficients to have some conditional independence property for the solution.

Which kind of conditional independence property should be expected? The classical Markov Process property will not hold in general because a random variable X_t can hardly make independent the past and the future of the process, given that the first and last variables are linked by the boundary condition.

It turns out that the relevant property to study is the Markov Field property. This is the natural Markov property for random fields, and therefore it is more clearly formulated with a general parameter set:

A random field $\{X_t,\ t \in T\}$, with $T \subset \mathbb{R}^k$, is a *Markov Field* (M.F., for short) if and only if for every Borel and bounded set D (with $\overline{D} \subset T$), the families of random variables $\{X_t,\ t \in \overline{D}\}$ and $\{X_t,\ t \in \overline{D^c}\}$ are conditionally independent given $\{X_t,\ t \in \partial D\}$. Here ∂D denotes the boundary of D. It is enough to check this property for open sets D. Translated to the one parameter case, with $T = [0,1]$, the property can be stated as follows: $\{X_t,\ t \in [0,1]\}$ is a Markov Field if and only if the families $\{X_u,\ u \in [s,t]\}$ and $\{X_u,\ u \in]s,t[^c\}$ are independent given X_s and X_t. It is obvious that every Markov Process is a Markov Field, but the converse is not true.

Let us now take a look to the particular equations studied so far, and the kind of results obtained concerning the Markov Field property. Our aim here is only to sketch these results. We refer the reader to the original references for the precise statements. Particularly, we remark that some technical hypothesis on the coefficients of the equations are needed, first of all, to obtain existence and uniqueness theorems, and then, to characterize the Markov Field property.

A) **First order equations**: Equations like (2.1) are first order equations. Within this setting, the first work was done by Ocone and Pardoux [19] (1989), who considered *linear equations*:

$$\left.\begin{aligned} dX_t = (AX_t + a(t))\,dt + \sum_{i=1}^{k}(B_iX_t + b_i(t)) \circ dW_t^i \quad , \quad 0 \le t \le 1 \\ F_0X_0 + F_1X_1 = F \end{aligned}\right\} \tag{2.3}$$

(affine drift, diffusion coefficient and boundary condition), where A, $B_1, \dots,$ B_k, F_0, F_1 are $d \times d$-matrices of constants, $F \in I\!R^d$, and $a(t), b_1(t), \dots, b_k(t)$ are d-dimensional processes.

Their main result states that each of the following are sufficient conditions to have a M.F.

a) $B_1 = \cdots = B_k = 0$ (Gaussian case).
b) $a = b_1 = \cdots = b_k = 0$ and $\Phi_t \cdot \Phi_s^{-1}$ is a diagonal matrix, $\forall t, s$, where Φ_t is the matrix solution of

$$\left.\begin{aligned} d\Phi_t = A\Phi_t\,dt + \sum_{i=1}^{k} B_i\Phi_t \circ dW_t^i \quad , \quad 0 \le t \le 1 \\ \Phi_0 = I \end{aligned}\right\} \tag{2.4}$$

Notice that, in particular, in dimension one and with linear drift and diffusion, the solution is always a M.F. Further results can be given for special forms of the boundary condition (see [19]).

Nualart and Pardoux [16] considered the non-linear equation

$$\left.\begin{aligned} dX_t = f(X_t)\,dt + \sigma\,dW_t \quad , \quad 0 \le t \le 1 \\ h(X_0, X_1) = 0 \end{aligned}\right\} \tag{2.5}$$

and proved that, in dimension 1 ($d = 1$), X is a M.F. iff f is affine. For $d > 1$, they showed examples with nonlinear f for which X is a M.F. (or even a Markov Process) and others where X is not a M.F. With certain special structures of the function f one can recover dichotomy results (see Ferrante [8] and Ferrante–Nualart [9]). ing

Donati-Martin [5] took the next step considering a linear diffusion coefficient:

$$\left.\begin{aligned} dX_t = f(X_t)\,dt + \sigma X_t \circ dW_t \quad , \quad 0 \le t \le 1 \\ F_0X_0 + F_1X_1 = F \end{aligned}\right\} \tag{2.6}$$

In dimension 1, the solution is a M.F. iff f takes the form $f(x) = Ax + Bx\log x$, with $|B| < 1$.

B) **Second order equations**: In [17], Nualart and Pardoux studied the following second order stochastic differential equation in dimension one, with Dirichlet type boundary conditions:

$$\left.\begin{array}{c}\ddot{X}_t = f(X_t, \dot{X}_t) + \dot{W}_t \quad , \quad 0 \le t \le 1 \\ X_0 = a,\ X_1 = b\end{array}\right\} \tag{2.7}$$

A solution to (2.7) will be a $\mathcal{C}^1$ process verifying

$$\left.\begin{array}{c}\dot{X}_t = \dot{X}_0 + \int_0^t f(X_s, \dot{X}_s)\,ds + W_t \quad , \quad 0 \le t \le 1 \\ X_0 = a,\ X_1 = b\end{array}\right\} \tag{2.8}$$

In this problem, we cannot expect to have any type of conditional independence property for X, because in a $\mathcal{C}^1$ process the positions X_t do not keep enough information to make independent the past and the future or the interior and exterior of an interval.

However, the two-dimensional stochastic process $\{(X_t, \dot{X}_t),\ t \in [0,1]\}$ is a M.F. iff f is affine. The same result is true if we change Dirichlet to Neumann boundary conditions (see Nualart [14]).

C) **Partial differential equations**: The following parabolic stochastic partial differential equation with periodicity conditions has been considered by Nualart and Pardoux [18]:

$$\left.\begin{array}{rl}\dfrac{\partial X_{t,y}}{\partial t} - \dfrac{\partial^2 X_{t,y}}{\partial y^2} = f(X_{t,y}) + \dfrac{\partial^2 W_{t,y}}{\partial t \partial y} & , \quad (t,y) \in [0,1]^2 \\ X(t,0) = X(t,1) = 0 & , \quad t \in [0,1] \\ X(0,y) = X(1,y) & , \quad y \in [0,1]\end{array}\right\} \tag{2.9}$$

where $\dfrac{\partial^2 W}{\partial t \partial y}$ is a space-time White Noise. The condition f affine is again necessary and sufficient for the Markov Field property of the $C_{0,0}([0,1])$-valued process X_t.

Donati-Martin [6], [7], considered the elliptic equation with Dirichlet boundary condition:

$$\left.\begin{array}{c}\Delta X_t = f(X_t) + \dot{W}_t \quad , \quad t \in T \\ X_{|\delta T} = 0\end{array}\right\} \tag{2.10}$$

where Δ is the Laplacian, T is a bounded domain of $\mathbb{R}^k$ $(k \le 3)$, and $\dot{W}$ represents a White Noise in $\mathbb{R}^k$. Here, the condition f affine is equivalent to a slightly weaker conditional independence property: For any Borel and bounded set D (with $\overline{D} \subset U \subset T$, for some open set U), the σ-fields $\sigma\{X_t,\ t \in \overline{D}\}$ and $\sigma\{X_t,\ t \in \overline{D^c}\}$ are conditionally independent given the σ-field $\bigcap_{\varepsilon>0} \sigma\{X_t,\ t \in (\partial D)_\varepsilon\}$, where $(\partial D)_\varepsilon$ denotes an ε-neighbourhood of ∂D. X is said to be a *Germ Markov Field* (G.M.F.). □

All these results concerning nonlinear equations have been obtained using a common method, which is based in an argument of change of measure in Wiener space. Our aim is to explain this method, and to illustrate it with some examples. In a non-anticipating context, a change of measure for s.d.e. would rely in the celebrated (Cameron–Martin–Maruyama)–Girsanov Theorem. In our case, an extended version of this theorem, essentially due to Ramer [20] and Kusuoka [12], and allowing for anticipating transformations, is the basic tool to use.

3. Idea of the change of measure method

The idea of the change of measure to study nonlinear anticipating s.d.e. is analogous to that of the classical Girsanov theorem for non-anticipating ones. Starting with a nonlinear equation on a probability space $(\Omega, \mathcal{F}, P)$, with solution process X_t, one considers another measure Q on $(\Omega, \mathcal{F})$, and a linear equation conveniently related with the original one. These measure and linear equation should be chosen in such a way that the law of the solution Y_t under Q coincide with the law of X_t under P.

Then, anything we can prove concerning the law of Y_t under Q produces automatically the same result for the law of X_t under P, which is the process we are interested in. In other words, we switch to a simpler process (possibly in explicit form), at the price of dealing with a more complicated measure, given by its Radon–Nikodým derivative with respect to P.

4. Some elements of analysis in Wiener space

Let (B, H, P) be a Wiener space. That is, H is an infinite-dimensional real separable Hilbert space, equipped with the Gauss cylinder measure μ, B is the completion of H with respect to a measurable norm, and P is the extension of μ to a measure on the Borel σ-field of B. P is called the *Wiener measure* on B. On the other hand, given a Banach space B and a Gaussian centered measure P on B, with $\operatorname{supp} P = B$, there exists a unique Hilbert space $H \subset B$ such that (B, H, P) is a Wiener space. We will call H the *Cameron–Martin space* relative to B and P. See for example Kuo [11] for details.

This is the definition of an *abstract Wiener space*. The *classical Wiener space* is the particular instance in which we take as B the space $C_0([0,1])$ of continuous functions vanishing at zero with the supremum norm, H is the subspace of functions with derivatives in $L^2([0,1])$, and P is the measure induced by a one-dimensional standard Wiener process.

Many interesting mappings and functionals on Wiener space, such as solutions to s.d.e., are not Fréchet differentiable in general; therefore the classical infinite-dimensional calculus is of little help. For this reason, several infinite-dimensional calculi well adapted to these functionals have been introduced. We recall here the definition of derivation and other basic features of the Malliavin infinite-dimensional calculus.

Let E be a real separable Hilbert space, $F\colon B \to E$ a mapping and $\omega_0 \in B$. We say that F is *H-differentiable* at ω_0 iff there exists $\nabla F(\omega_0) \in \mathcal{L}(H;E)$ (a linear continuous mapping from H into E) such that

$$\lim_{\substack{h\in H \\ ||h||_H \to 0}} \frac{||F(\omega_0+h) - F(\omega_0) - [\nabla F(\omega_0)](h)||_E}{||h||_H} = 0 \quad .$$

Clearly, if F is Fréchet differentiable at ω_0, then F is H-differentiable at ω_0 and $\nabla F(\omega_0)$ coincides with the Fréchet differential restricted to H.

A *smooth E-valued cylinder functional on B* is a mapping $F\colon B \to E$ of the form

$$F(\omega) = \sum_{j=1}^{m} f_j((\ell_1,\omega),\dots,(\ell_n,\omega))e_j \quad ,$$

where $\ell_1,\dots,\ell_n \in B^*$ (the topological dual of B), $e_j \in E$, and f_j are C^∞ functions on $\mathbb{R}^n$ with polynomial growth, together with all their derivatives. Denote by $\mathcal{S}(E)$ the set of these functionals.

Since an element $F \in \mathcal{S}(E)$ is clearly Fréchet differentiable on B, its H-differential exists for every ω and

$$\nabla F(\omega) = \sum_{j=1}^{m}\sum_{i=1}^{n} \partial_i f_j((\ell_1,\omega),\dots,(\ell_n,\omega))\ell_i \otimes e_j \quad ,$$

considered as an element of $\mathcal{L}(H;E)$. Of course, $(\ell_i \otimes e_j, h)$ means $(\ell_i, h)\cdot e_j$, and is an element of the algebraic tensor product $H \otimes E$ (after identification of H and H^*). Its completion by the inner product $\langle \ell \otimes e, \ell' \otimes e' \rangle := \langle \ell, \ell' \rangle_H \cdot \langle e, e' \rangle_E$ is the space of Hilbert–Schmidt operators from H to E, which we will denote by the same symbol $H \otimes E$.

Up to this point, we have only taken into account the topological structure of B. Now, using the measure P, one can prove that $\forall F \in \mathcal{S}(E)$, $\forall p \geq 1$, $F \in L^p(B;E)$ and $\nabla F \in L^p(B; H \otimes E)$, and that $\mathcal{S}(E)$ is dense in $L^p(B;E)$ (see Ikeda and Watanabe [13], Remark 8.2). Moreover, the mapping

$$\nabla\colon L^p(B;E) \longrightarrow L^p(B;H\otimes E) \quad ,$$

with domain $\mathcal{S}(E)$, is closable. Denoting by $I\!D^{1,p}(E)$ the closure of $\mathcal{S}(E)$ under the graph norm

$$||F||_{I\!D^{1,p}(E)} := ||F||_{L^p(B;E)} + ||\nabla F||_{L^p(B;H\otimes E)} \quad ,$$

we obtain a continuous mapping $\nabla: I\!D^{1,p}(E) \to L^p(B; H \otimes E)$, called the *gradient operator*. Recursively, one can define higher order gradient operators ∇^k ($\nabla^k F$ will be an element of $L^p(B; H^{\otimes k} \otimes E)$) and obtain the Sobolev spaces $I\!D^{k,p}(E)$.

The operator ∇ is local in the following sense: If, for some measurable set A, $F: B \to E$ verifies $F(\omega) = 0$, for a.a. $\omega \in A$, then $\nabla F: B \to H \otimes E$ verifies the same property. This fact justifies the following definition: The random variable $F: B \to E$ belongs to $I\!D^{k,p}_{\text{loc}}(E)$ iff there exist a sequence $\{B_n\}_{n\in I\!N}$ of measurable sets converging to B and a sequence $\{F_n\}_{n\in I\!N}$ of elements of $I\!D^{k,p}(E)$ such that $F_n = F$ on B_n. For $F \in I\!D^{k,p}_{\text{loc}}(E)$, the gradient ∇F is defined as $\nabla F(\omega) = \nabla F_n(\omega)$, if $\omega \in B_n$.

The following different concept of differentiability will be used in the Theorem below.

Definition. Let $F: B \to H$ be a random variable with values in the Cameron–Martin space H. The mapping F is H–$\mathcal{C}^1$ if for all $\omega \in B$, there exists a Hilbert–Schmidt operator $\mathcal{K}(\omega)$ such that

1) $||F(\omega + h) - F(\omega) - [\mathcal{K}(\omega)](h)||_H = o(||h||_H)$, as $||h||_H \to 0$, a.s.

2) The mapping $h \mapsto \mathcal{K}(\omega + h)$ from H to $H \otimes H$ is continuous, a.s. □

If F is H–$\mathcal{C}^1$, then $F \in I\!D^{1,2}_{\text{loc}}(H)$ (see Kusuoka [12] or Nualart [15]). On the other hand, $F \in I\!D^{1,2}_{\text{loc}}(H)$ implies that $\nabla F(\omega)$ is Hilbert–Schmidt, a.s. Therefore, $\nabla F(\omega)$ is the only candidate for the operator $\mathcal{K}(\omega)$ in the definition.

For any Hilbert-Schmidt operator $\mathcal{K}$ on a Hilbert space H, its Carleman–Fredholm determinant, denoted by $\det_2(I_H + \mathcal{K})$, is defined by

$$\det{}_2(I_H + \mathcal{K}) := \prod_{i=1}^{\infty}(1 + \lambda_i)e^{-\lambda_i} \quad ,$$

where $\{\lambda_i\}_{i=1}^{\infty}$ is the family of (complex) eigenvalues of $\mathcal{K}$, counted with their multiplicity. For the properties of this quantity and its role in the theory of integral equations see, for instance, Cochran [4].

The Ramer–Kusuoka Theorem can be stated as follows:

Theorem (Ramer [20], Kusuoka [12]).
Let $F: B \to H$ be an H–$\mathcal{C}^1$ map. Assume:

a) The transformation $T: B \to B$ given by $T(\omega) = \omega + F(\omega)$ is bijective.

b) The operator $I_H + \nabla F(\omega): H \to H$ is invertible, a.s.

Then:

The measure $Q := P \circ T$ (the image probability of the Wiener measure by T^{-1}) is equivalent to P and

$$\frac{dQ}{dP}(\omega) = \big|\det_2(I_H + \nabla F(\omega))\big| \exp\big\{ -(\nabla^* F)(\omega) - \frac{1}{2}||F(\omega)||_H^2 \big\} \quad ,$$

where ∇^* is the adjoint of the gradient operator, considered here as an unbounded operator $L^2(B) \to L^2(B; H)$. □

Remark. ∇^* enjoys a local property which is similar to that of ∇, and ensures that, for $F \in I\!D^{1,2}_{\text{loc}}(H)$, $\nabla^* F$ is well-defined. □

Remark. There exist stronger versions of this theorem, but we will not make use of them. Particularly, Üstünel and Zakai ([23] [24]) have obtained representations for the density of Q without hypothesis *a)* and with less regularity on F. □

Sometimes it is useful, for the purpose of representation, to realize the Cameron–Martin space H as an L^2 space. Let $(T, \mathcal{B}, \mu)$ be a separable measure space. Denote $\tilde{H} = L^2(T, \mathcal{B}, \mu)$, and let $i: \tilde{H} \to H$ be an isomorphism. Given a random variable $F: B \to H^{\otimes n}$ (for some $n = 0, 1, 2, \dots$), define $G: B \to \tilde{H}^{\otimes n}$ by the equality $F = i^{\otimes n} \circ G$, where $i^{\otimes n}$ is the natural isomorphism between $\tilde{H}^{\otimes n}$ and $H^{\otimes n}$ induced by i. Define, also,

$$DG(\omega) := (i^{\otimes n})^{-1} \circ \nabla F(\omega) \circ i \quad \in \tilde{H}^{\otimes n+1} \quad .$$

If $F \in I\!D^{1,2}(H^{\otimes n})$, then clearly $G \in I\!D^{1,2}(\tilde{H}^{\otimes n})$ and we have $DG \in L^2(B; \tilde{H}^{\otimes n+1}) \simeq L^2(B \times T; \tilde{H}^{\otimes n})$.

The operator

$$D: I\!D^{1,2}(\tilde{H}^{\otimes n}) \longrightarrow L^2(B \times T; \tilde{H}^{\otimes n}) \quad ,$$

which transforms random variables into processes (indexed by the elements of T), is called the *Malliavin derivative operator.*

The statement of the Ramer–Kusuoka Theorem can be translated, using the Malliavin derivative, into a form which is usually more amenable to computations. Let $F: B \to H$, as in the Theorem. Then $DG(\omega) = i^{-1} \circ \nabla F(\omega) \circ i$ is a Hilbert–Schmidt operator on $\tilde{H}$, which has the same eigenvalues of $\nabla F(\omega)$, and the invertibility of $I_H + \nabla F(\omega)$ is equivalent to that of $I_{\tilde{H}} + DG(\omega)$.

Assume now $v: B \to I\!R$, $v \in L^2(B)$. In this case, $Dv(\omega) = \nabla v(\omega) \circ i$. If we denote by j the isomorphism between $L^2(B; \tilde{H})$ and $L^2(B; H)$ given by $j(F)(\omega) = i(F(\omega))$, we can write $D = j^{-1} \circ \nabla$ (as unbounded operators on $L^2(B)$) and the relation $D^* = \nabla^* \circ j$ holds for the adjoints. The adjoint of the Malliavin derivative operator D is usually denoted by δ and is called the *Skorohod integral.* We have then, for $F: B \to H$ and $F = i \circ G$,

$$\delta G = \nabla^* F \quad .$$

The formula for the density of Q with respect to P becomes

$$(4.1) \qquad \frac{dQ}{dP}(\omega) = \left| \det{}_2(I_{\tilde{H}} + DG(\omega)) \right| \exp\left\{ -(\delta G)(\omega) - \frac{1}{2} ||G(\omega)||^2_{\tilde{H}} \right\} \quad .$$

It is convenient to keep in mind both interpretations of $DG(\omega)$. In fact, the process $\{D_t G(\omega),\ t \in T\}$ is the kernel of the integral operator $DG(\omega)$:

$$[DG(\omega)](h) = \int_T D_t G(\omega) h(t)\, dt, \quad h \in \tilde{H} \quad .$$

In the case of the classical Wiener space, taking $T = [0,1]$ with the Lebesgue measure, we have $i(h) = \int_0^{\cdot} h(t)\, dt$.

5. Transformations. Existence and Uniqueness of solutions

As stated before, transformations in Wiener space provide a natural method to achieve existence and uniqueness results for nonlinear s.d.e. We are going to describe here this procedure, and to apply it to some concrete examples. Another natural approach involves the use of stochastic flows (see, for instance, [5], Theorems 3.1 and 4.1).

Suppose we have a nonlinear equation of the following form:

$$(\mathrm{N}) \qquad \left. \begin{aligned} p(D)X_t &= f(X_t) + \sigma(X_t) \circ \dot{W}_t \quad , \quad t \in T \subset I\!R^k \\ h(X_{|\delta T}) &= 0 \end{aligned} \right\}$$

where $p(D)$ is a linear differential operator with constant coefficients, and we assume that σ is linear or constant. For simplicity of notation we make f, σ and h depend only on X and not on any of its derivatives (cf. equation (2.7)) or the time parameter t, but this is also possible. We assume also that the dimension is equal to one.

$\dot{W}_t$ represents a White Noise. That means: $\dot{W}$ is a centered Gaussian family $\{\dot{W}(A),\ A \in \mathcal{B}(I\!R^k)\}$ with covariance $E[\dot{W}(A)\dot{W}(B)]$ equal to the Lebesgue measure of $A \cap B$. Its (random) distribution function W_t (normalized with $W_0 = 0$) is a standard Wiener process with k-dimensional parameter. We can (and shall) assume that the probability space $(\Omega, \mathcal{F}, P)$ in which we are working is the canonical space of the Wiener process. Then $W_t(\omega) = \omega(t),\ t \in T$.

Step 1: We can associate to (N) the linear equation

$$(\mathrm{L}) \qquad \left. \begin{aligned} p(D)Y_t &= \alpha Y_t + \sigma(Y_t) \circ \dot{W}_t \quad , \quad t \in T \\ h(Y_{|\delta T}) &= 0 \end{aligned} \right\}$$

for some convenient $\alpha \in I\!R$ (which very often can be taken to be zero). Usually, it is not difficult to find some explicit expression for the solution Y_t.

Step 2: Denote by Σ the set of trajectories of the process Y_t. We must identify this set and check that the mapping from Ω into Σ, defined by $\omega \mapsto Y(\omega)$, is bijective.

Step 3: Define a transformation $T: \Omega \to \Omega$ by $T(\omega) = \omega + F(\omega)$, with $F: \Omega \to H$. We must choose F in order to have the following:

a) If $Z: \Omega \to \Omega$ satisfies $T(Z(\omega)) = \omega$, then $X(\omega) := Y(Z(\omega))$ solves (N).

b) If X solves (N), then there exists such a Z.

F is usually found by inspection or by a formal manipulation of (N) and (L). The following fact is immediate:

Proposition. If T is exhaustive, there exists a solution to (N) whose paths belong to Σ. If, moreover, T is injective, the solution is unique, *within* the class of processes with paths in Σ. $\square$

Example 1. Let us consider first an equation of the type (2.5):

$$\left.\begin{array}{c} \dot{X}_t = f(X_t) + \sigma \dot{W}_t \quad , \quad 0 \le t \le 1 \\ h(X_0, X_1) = 0 \end{array}\right\} \tag{5.1}$$

We assume here $d = 1$ and $\sigma > 0$. The linearized equation (L) is

$$\left.\begin{array}{c} \dot{Y}_t = \alpha Y_t + \sigma \dot{W}_t \quad , \quad 0 \le t \le 1 \\ h(Y_0, Y_1) = 0 \end{array}\right\} \tag{5.2}$$

The solution with initial condition Y_0 can be easily computed and (5.2) becomes equivalent to the system

$$\left.\begin{array}{c} Y_t = e^{\alpha t}\big(Y_0 + \int_0^t \sigma e^{-\alpha s}\, dW_s\big) \quad , \quad 0 \le t \le 1 \\ h\big(Y_0, e^{\alpha}\big(Y_0 + \int_0^1 \sigma e^{-\alpha s}\, dW_s\big)\big) = 0 \end{array}\right\} \tag{5.3}$$

Taking into account that the random variable $\int_0^1 e^{-\alpha s}\, dW_s$ is absolutely continuous and has the whole real line as support, (5.3) will have a solution iff $\forall z \in \mathbb{R}$ (a.e.), $h(y, e^{\alpha} y + z) = 0$ has a solution $y = g(z)$. And this happens when $h(X_0, X_1) = 0$ can be written as $X_0 = g(X_1 - e^{\alpha} X_0)$. Thus, this should be the case to take advantage of considering equation (L). For example, choosing α properly, the case of affine boundary conditions ($aX_0 + bX_1 + c = 0$) is fully covered. But observe also that with periodicity conditions ($X_0 = X_1$) we cannot take the simplest equation ($\alpha = 0$).

The solution to (L) is then

$$Y_t = e^{\alpha t}\big[g\big(e^{\alpha} \int_0^1 \sigma e^{-\alpha s}\, dW_s\big) + \int_0^t \sigma e^{-\alpha s}\, dW_s\big] \quad .$$

It is immediate to prove that $\Sigma = \{\xi \in C([0,1]) : \xi_0 = g(\xi_1 - e^{\alpha}\xi_0)\}$ and that $Y\colon \Omega \to \Sigma$ is a bijection. This follows from the fact that any continuous function vanishing at zero is a trajectory of $\int_0^t e^{-\alpha s}\, dW_s$ (this integral has a sense pathwise).

To find the random variable F notice first that $T(\omega) = \omega + F(\omega)$ implies $T^{-1}(\omega) = \omega - F(T^{-1}(\omega))$. Then, using that $Y(T^{-1}(\omega))$ solves (N) and, at the same time, solves (L) when $W(\omega)$ is substituted by $W(T^{-1}(\omega))$, we find

$$F(\omega)_t = \frac{1}{\sigma}\int_0^t \big(\alpha Y_s(\omega) - f(Y_s(\omega))\big)\, ds \quad .$$

The bijectivity of T is true under some monotonicity or Lipschitz conditions on f (see Nualart and Pardoux [16], Propositions 2.2 and 2.5 and subsequent remarks).

Example 2. Consider the second order equation

$$\ddot{X}_t = f(X_t) + \dot{W}_t \quad . \tag{5.4}$$

The solution of the corresponding equation (L)

$$\ddot{Y}_t = \alpha Y_t + \dot{W}_t \quad , \tag{5.5}$$

with initial conditions Y_0 and $\dot{Y}_0$, is, for $\alpha > 0$, and putting $\lambda = \sqrt{\alpha}$,

$$Y_t = Y_0 \cosh \lambda t + \dot{Y}_0 \frac{1}{\lambda} \sinh \lambda t + \int_0^t \cosh\big(\lambda(t-s)\big) W_s\, ds \quad .$$

For $\alpha < 0$, we get the same expression with the hyperbolic functions replaced by the corresponding trigonometric functions, and $\lambda = \sqrt{-\alpha}$. For $\alpha = 0$, we get simply

$$Y_t = Y_0 + \dot{Y}_0 t + \int_0^t W_s\, ds \quad .$$

A general boundary condition will have the form $h(X_0, \dot{X}_0, X_1, \dot{X}_1) = (0,0)$. As in Example 1, it is easy to find which form it must take (depending on α) to have a solution to (5.5). We would find that, for Dirichlet boundary conditions ($X_0 = 0$, $X_1 = 0$), one can take $\alpha = 0$, which is convenient to simplify later computations. But for Neumann ($\dot{X}_0 = 0$, $\dot{X}_1 = 0$) or periodicity ($X_0 = X_1$, $\dot{X}_0 = \dot{X}_1$) conditions, one must take $\alpha \neq 0$.

We must define

$$F(\omega)_t = \int_0^t \big(\alpha Y_s(\omega) - f(Y_s(\omega))\big)\, ds \quad ,$$

and it can be shown that if $y \mapsto \alpha y - f(y)$ is locally Lipschitz, nonincreasing, and with linear growth, then T is bijective.

Whatever boundary condition we prescribe, Y will be a bijection between Ω and the set Σ of $\mathcal{C}^1$ functions verifying the boundary conditions. This can be checked as in Example 1. The following example behaves differently concerning this point.

Example 3. Let us add a linear diffusion coefficient, depending only on $\dot{X}_t$, to the previous equation,

$$\ddot{X}_t = f(X_t) + \sigma \dot{X}_t \circ \dot{W}_t \quad , \tag{5.6}$$

and impose the boundary conditions $X_0 = 0$, $X_1 = 1$. Then, taking $\alpha = 0$ we can solve the associated equation (L) and get the solution

$$Y_t = \left(\int_0^1 e^{\sigma W_s} \, ds \right)^{-1} \cdot \int_0^t e^{\sigma W_s} \, ds \quad .$$

In this case, the set Σ consists of all $\mathcal{C}^1([0,1])$ functions ξ with $\xi' > 0$ in $[0,1]$ and $\xi_0 = 0$, $\xi_1 = 1$. Therefore, one is bound to prove existence and uniqueness of solutions to (5.6) within the class of processes with such trajectories. We must take

$$F(\omega)_t = \int_0^t -\frac{f(Y_s(\omega))}{\sigma \dot{Y}_s(\omega)} \, ds \quad .$$

This equation will be studied by Alabert and Nualart in a forthcoming paper. We remark that if we impose $X_0 = 0$, $X_1 = 0$, the solution to (L) is $Y_t \equiv 0$, and the bijectivity of $Y \colon \Omega \to \Sigma$ fails.

Example 4. Consider the partial differential equation (2.10), and assume ∂T is a smooth hypersurface. We must precise first what is meant by a solution to this equation. If the righthand side were a continuous function g, the solution would have the representation

$$X_t = \int_T -K(t,s) g(s) \, ds$$

for a symmetric kernel K. It is natural then to define a solution to (2.10) as a continuous process X_t verifying the integral equation

$$X_t = \int_T -K(t,s) f(X_s) \, ds + \int_T -K(t,s) \, \dot{W}(ds) \quad . \tag{5.7}$$

This concept of solution coincides, on the other hand, with the one obtained thinking of (2.10) as an equation between distributions (see Buckdahn and Pardoux [3]).

Notice that $V_t = \int_T -K(t,s)\,\dot{W}(ds)$ defines a Gaussian measure $\dot{V}$ on the Banach space B of continuous functions on T vanishing at ∂T. We can substitute this term in (5.7). The auxiliary linear equation

$$\left.\begin{array}{ll} \Delta Y_t = \dot{W}_t \quad , & t \in T \\ Y_{|\delta T} = 0 & \end{array}\right\} \tag{5.8}$$

has precisely the solution $Y_t = V_t$. Therefore, in this case, we can take $\Omega = \Sigma = B$, and Y is the identity mapping.

The restriction $k \leq 3$ is due to the fact that otherwise $K(t,s)$ is not square-integrable and the linear equation fails to have a solution (see Rozanov [21]).

F must be defined as

$$F(v)_t = \int_T K(t,s) f(v(s))\,ds \quad , \quad v \in B \quad , \tag{5.9}$$

and the bijectivity of T is obtained under the condition that f is continuous and non-decreasing (Donati-Martin [6], Lemma 3.1).

6. Change of measure and Markov properties

In this Section we will describe the use of the transformations $T(\omega) = \omega + F(\omega)$, with $F\colon \Omega \to H$, to obtain conditions for a Markov type property to hold. Recall that we choose T in order to have $X_t = T^{-1}(Y)_t$, for X_t and Y_t solving equations (N) and (L), respectively. Therefore, defining $Q = P \circ T$, the law of Y under Q coincides with the law of X under P.

We decompose the procedure in several steps, as in the previous Section, to clarify the exposition. The objective is to obtain an equivalence between the Markov Field property and a measurability condition which depends only on the structure of the Carleman–Fredholm determinant $\det_2(I_{\tilde{H}} + DG(\omega))$. It is hoped that this measurability condition could be then translated into an analytical condition on the coefficients of the equation.

We assume we have performed steps 1 to 3 of Section 5, and that T is bijective.

Step 4: Check that, in the case of the linearized equation (L), the process we are interested in (the solution Y_t, the couple $(Y_t, \dot{Y}_t)$, or whatever) is a M.F. under the original probability P. Since possibly the process is in explicit form, there should be no difficulties in proving this fact by some direct method. Ocone–Pardoux [19], Rozanov [21], and Russek [22], for instance, give general results in this direction.

Step 5: Verify the hypothesis of the Ramer–Kusuoka Theorem and obtain $J(\omega) := \dfrac{dQ}{dP}(\omega)$. This involves, in principle, the computation of $\det_2(I_{\tilde{H}} +$

$DG(\omega)$), which is always rather cumbersome, but sometimes can be avoided, as in Example 4 below.

Step 6: Let us introduce the following notation: Given a subset D of the parameter space, such that $\overline{D} \subset T$, set

$$\mathcal{F}_D^i := \sigma\{Y_t,\ t \in \overline{D}\}, \quad \mathcal{F}_D^e := \sigma\{Y_t,\ t \in \overline{D^c}\}, \quad \mathcal{F}_D^b := \sigma\{Y_t,\ t \in \partial D\}.$$

We will write E_P and E_Q to denote the expectation taken with respect to probabilities P and Q, respectively. The following Lemma is an immediate consequence of the definition of conditional expectation and the Radon-Nikodým Theorem.

Lemma 1. For every random variable ξ, $\mathcal{F}_D^e$-measurable and Q-integrable,

$$\mathrm{E}_Q\left[\xi/\mathcal{F}_D^i\right] = \frac{\mathrm{E}_P\left[\xi J/\mathcal{F}_D^i\right]}{\mathrm{E}_P\left[J/\mathcal{F}_D^i\right]} \qquad \square \tag{6.1}$$

The quotient is indeed well defined because the invertibility of $I_{\tilde{H}} + DG(\omega)$, which is assumed a.s., is equivalent to $\det_2(I_{\tilde{H}} + DG(\omega)) \neq 0$. Therefore, $J > 0$, a.s.

Now we have the equivalencies: X is a M.F. under P iff Y is a M.F. under Q iff $\forall D$ and $\forall \xi$, $\mathcal{F}_D^e$-measurable, $\mathrm{E}_Q\left[\xi/\mathcal{F}_D^i\right]$ is $\mathcal{F}_D^b$-measurable, iff the same property is true for the quotient in (6.1).

Thus, we get a measurability condition formulated again in terms of the original probability P, the σ-fields generated by the process Y and (at the cost of) the density J.

Step 7: Suppose at first that $J(\omega)$ can be written, for every Borel set D, as $J = L_D^e L_D^i$, with L_D^e and L_D^i random variables $\mathcal{F}_D^e$ and $\mathcal{F}_D^i$-measurable, respectively. Then, the Markov Field property holds: Indeed, using that Y is a M.F. under P (step 4), and this factorization for J, the quotient in (6.1) is clearly $\mathcal{F}_D^b$-measurable.

In all examples known, the exponential part of (4.1) can be factorized in this way, and it only remains to consider the Carleman–Fredholm determinant. In the adapted case, $\det_2(I_{\tilde{H}} + DG(\omega))$ is always equal to 1 and the factorization holds. Therefore, in the classical Wiener space, J reduces to Girsanov formula (recall that for adapted G, the Skorohod integral coincides with the Itô integral). This fact is literally "trivial": Zakai and Zeitouni show in [25] three different arguments.

Assume then that there is a partial factorization $J = Z L_D^e L_D^i$, for some Z. This simplifies the measurability condition:

$$\frac{\mathrm{E}_P\left[\xi J/\mathcal{F}_D^i\right]}{\mathrm{E}_P\left[J/\mathcal{F}_D^i\right]} = \frac{\mathrm{E}_P\left[\xi Z L_D^e/\mathcal{F}_D^i\right]}{\mathrm{E}_P\left[Z L_D^e/\mathcal{F}_D^i\right]} \quad ,$$

and taking $\xi = \eta \cdot (L_D^e)^{-1}$, with η an $\mathcal{F}_D^e$-measurable random variable, we get that the Markov Field property is equivalent to the fact that

$$\Lambda_\eta := \frac{\mathrm{E}_P\left[\eta Z/\mathcal{F}_D^i\right]}{\mathrm{E}_P\left[Z/\mathcal{F}_D^i\right]} \quad \text{is } \mathcal{F}_D^b\text{-measurable.} \quad \square \tag{6.2}$$

From this point onwards, the objective is to translate this characterization into an analytical one. The way to do this depends very much on the concrete problem under study, but in most cases the arguments employed make a fundamental use of the following Lemma:

Lemma 2. Let $F \in \mathbb{D}^{1,2}_{\mathrm{loc}}$, and M a closed subspace of $L^2(T)$. Denote by $\mathcal{F}_M$ the σ-field generated by the Gaussian random variables $\{\int_T h\,dW,\ h \in M\}$. Let $A \in \mathcal{F}_M$ and assume that $1_A F$ is $\mathcal{F}_M$-measurable. Then $DF(\omega) \in M$, for $\omega \in A$, a.s. $\square$

For the proof, see for instance [15]. We illustrate steps 4 to 7 above with the examples that follow. In the second one, moreover, we sketch the derivation of the final analytical characterization, without going into details.

Finally, we notice that for the Germ Markov Field property, everything in this Section remains valid after changing the definition of $\mathcal{F}_D^b$.

Example 2 *(Continued)*. Consider again the second order equation (5.4), with Neumann boundary conditions ($\dot{X}_0 = 0, \dot{X}_1 = 0$). Take for instance $\alpha = 1$ in equation (L).

We have the solution

$$Y_t = -\frac{W_1 + \int_0^1 \sinh(1-s)W_s\,ds}{\sinh 1} \cdot \cosh t + \int_0^t \cosh(t-s)W_s\,ds \quad .$$

The process $(Y_t, \dot{Y}_t)$ is a M.F. (in fact, a Markov Process). This can be seen computing, for $t > s$, $\mathrm{E}\left[\psi(Y_t,\dot{Y}_t)/(Y_r,\dot{Y}_r), 0 \le r \le s\right]$ for a bounded measurable ψ by means of a regular version of the conditional probability. The result is a function of Y_s and $\dot{Y}_s$.

If $\bar{f}(y) := \alpha y - f(y)$, besides the conditions stated in Section 5, is of class $\mathcal{C}^2$, the hypothesis of the Ramer–Kusuoka Theorem are satisfied:

F is H–$\mathcal{C}^1$ because in fact it is a Fréchet continuously differentiable mapping $\Omega \to H$. We have $G(\omega)_s := (i^{-1} \circ F)(\omega)_s = Y_s(\omega) - f\big(Y_s(\omega)\big)$ (cf. Section 4). The derivative operator satisfies a chain rule analogous to that of the ordinary derivative. This allows to compute the kernel of $DG(\omega)$:

$$D_t G(\omega)_s = \bar{f}'(Y_s) \cdot \left[-\frac{\cosh s}{\sinh 1} \cdot \cosh(1-t) - \sinh(1-t) \cdot 1_{\{t \ge s\}}\right] \quad .$$

From the Fredholm alternative, we have to check that the unique solution in $L^2([0,1])$ of the integral equation

$$h(s) + \int_0^1 D_t G(\omega)_s h(t)\,dt = 0 \tag{6.3}$$

is $h \equiv 0$.

Set $g(t) = \int_0^t \sinh(t-s)h(s)\,ds$. From (6.3), $g(t)$ must satisfy

$$g'(t) = g'(1) \cdot \int_0^t \Phi_{22}(t,s) \cdot \bar{f}'(Y_s) \cdot \frac{\cosh s}{\sinh 1}\,ds \quad ,$$

where $\Phi(t,s) = \Phi_t \cdot \Phi_s^{-1}$, and Φ_t is the solution to

$$\left.\begin{array}{c} d\Phi_t = M_t \Phi_t\,dt \quad , \quad 0 \le t \le 1 \\ \Phi_0 = I \end{array}\right\}$$

for $M_t = \begin{pmatrix} 0 & 1 \\ 1 - \bar{f}'(Y_t) & 0 \end{pmatrix}$.

The integrand is nonpositive, and we deduce $g'(1) = 0$, hence $h \equiv 0$. Therefore, we can apply the Theorem, and follow steps 6 and 7, to get the condition (6.2), with $Z = \det_2(I_{\tilde{H}} + DG(\omega))$. The same computations can be done taking any $\alpha > 0$, and the following analytical condition is derived (we refer to Nualart [14] for the final computations):

Theorem: If f is C^2, with lineal growth, and $f' \ge \alpha$, for some $\alpha > 0$, then (5.4) has a unique solution X_t, and the process $\{(X_t, \dot{X}_t), 0 \le t \le 1\}$ is a M.F. iff $f'' = 0$.

Example 4 *(Continued).* In the previous Section we have set the Banach space $B = \{\omega \in C(\overline{T}) : \omega_{|\delta T} = 0\}$, with the supremum norm. Let μ be the law of $V\colon \Omega \to B$, which is a Gaussian measure on B. Instead of the Cameron–Martin space H relative to B and μ, we will choose an L^2 space to work in. Take $\tilde{H} = L^2(T)$. The isomorphism $i\colon \tilde{H} \to H$ is given by

$$h \mapsto \int_T K(t,\cdot)h(t)\,dt \quad .$$

Indeed, $i\colon \tilde{H} \to B$ is continuous with a dense image, and moreover every $\ell \in B^*$ is a random variable normally distributed, with mean zero and variance $||\ell||_H$. This characterizes (B, H, μ) as an abstract Wiener space and can be checked by computing the characteristic function of ℓ.

Recall the definition of the random variable F in (5.9). Since $F(\omega) = i(f \circ \omega)$, we have $G(\omega) = f \circ \omega$. Assuming $f \in C^1$ and $f' > 0$, the hypothesis of the Ramer–Kusuoka Theorem are satisfied: It is immediate to compute $\nabla F(\omega)$ using Fréchet differentials:

$$[\nabla F(\omega)](h) = \int_T K(t,\cdot) f'(\omega_t) h(t)\,dt \quad .$$

Then, from the relation between $DG(\omega)$ and $\nabla F(\omega)$, we find the kernel

$$D_t G(\omega)_s = f'(\omega_s) K(t,s) \quad .$$

Moreover the mapping from $\tilde{H}$ into $L^2(T\times T)$ given by $h \mapsto f'(\omega_s+h_s)K(t,s)$ is obviously continuous.

We know that T is bijective. Finally, we need the invertibility of $I_{\tilde{H}} + DG(\omega)$. Equivalently, the integral equation

$$h_s + \int_T f'(\omega_s)K(t,s)h(t)\,dt = 0$$

must have $h \equiv 0$ as the unique solution in $L^2(T)$ (Fredholm alternative). Indeed, multiplying by $\frac{h(s)}{f'(\omega_s)}$ and integrating over T, we get

$$\int_T \frac{h^2(s)}{f'(\omega_s)}\,ds + \langle h, \mathcal{K}(h)\rangle_{\tilde{H}} = 0 \quad ,$$

where $\mathcal{K}(h) = \int_T K(t,\cdot)h(t)\,dt$. But it is known that $\langle h, \mathcal{K}(h)\rangle_{\tilde{H}} \geq 0$ (see, for instance, [3]). Both terms must be zero, from which $h \equiv 0$. Therefore, formula (4.1) holds true for the density $\frac{dQ}{dP}$.

The exponential part of (4.1) factorizes as explained in Step 7 (see [6]). Therefore, following steps 4 to 7, we arrive to the condition (6.2), with $\mathcal{F}_D^b = \bigcap_{\varepsilon>0} \sigma\{Y_t,\ t \in (\partial D)_\varepsilon\}$ and $Z = \det_2(I_{\tilde{H}} + DG(\omega))$.

In the sequel, we are going to avoid the references to the set D and the space $\tilde{H}$, to simplify notation. Set also $\mathcal{F}^{b,\varepsilon} = \sigma\{Y_t,\ t \in (\partial D)_\varepsilon\}$. Let us assume that $f \in \mathcal{C}^2$ and $f' > 0$. We want to prove that X is a G.M.F. iff $f'' \equiv 0$.

One of the implications is obvious: If $f'' \equiv 0$, then Z is deterministic and the factorization holds. Conversely, suppose X is a G.M.F. We notice that $Z \geq 0$, because the eigenvalues of the operator with kernel $f'(\omega_s)K(t,s)$ are nonnegative. This is also a consequence of $\langle h, \mathcal{K}(h)\rangle \geq 0$.

For every $\varepsilon > 0$, Λ_η is $\mathcal{F}^{b,\varepsilon}$-measurable. We can assume $\eta \geq 0$ and smooth (in the sense of Section 4). Then $\Lambda_\eta \in \mathbb{D}^{1,2}_{\text{loc}}$, and by Lemma 2 above,

$$D.\Lambda_\eta \in \operatorname{span}\{K(t,\cdot),\ t \in (\partial D)_\varepsilon\} \subset \tilde{H} \quad .$$

Let $\phi \in \mathcal{C}^\infty_{\text{comp}}(D - (\partial D)_\varepsilon)$. Then

$$\text{(6.3)} \qquad \langle \phi., \Delta D.\Lambda_\eta\rangle = 0 \quad \text{and} \quad \langle \phi., \Delta D.\eta\rangle = 0 \quad .$$

The operators Δ and D commute with the conditional expectations. Then, from (6.3), and using the chain rule for the operator D, we get

$$\mathrm{E}\left[{}^{Z}/_{\mathcal{F}^i}\right]\mathrm{E}\left[{}^{\eta\langle\phi., \Delta D.Z\rangle}/_{\mathcal{F}^i}\right] - \mathrm{E}\left[{}^{\eta Z}/_{\mathcal{F}^i}\right]\mathrm{E}\left[{}^{\langle\phi., \Delta D.Z\rangle}/_{\mathcal{F}^i}\right] = 0 \quad .$$

This implies that for every $\mathcal{F}^i$-measurable random variable ζ,

$$\mathrm{E}\left[\zeta\eta\,\mathrm{E}\left[{}^{Z}/_{\mathcal{F}^i}\right]\langle\phi., \Delta D.Z\rangle\right] = \mathrm{E}\left[\zeta\eta Z\,\mathrm{E}\left[{}^{\langle\phi., \Delta D.Z\rangle}/_{\mathcal{F}^i}\right]\right] \quad ,$$

which in turn gives

$$\mathrm{E}\left[{}^{Z}/_{\mathcal{F}^i}\right]\langle\phi.,\Delta D.Z\rangle = Z\,\mathrm{E}\left[{}^{\langle\phi.,\Delta D.Z\rangle}/_{\mathcal{F}^i}\right] \quad ,$$

and we deduce that

$$\frac{1}{Z}\langle\phi.,\Delta D.Z\rangle \quad \text{is } \mathcal{F}^i\text{-measurable.} \tag{6.4}$$

The Malliavin derivative of Z can be written

$$D_t Z = Z\cdot \mathrm{Tr}\left[(I+DG(\omega))^{-1}\circ(-DG(\omega))\circ D_t(DG(\omega))\right] \quad ,$$

where Tr is the trace of the nuclear operator in brackets. Denoting by L the kernel of the Hilbert–Schmidt operator $(I+DG(\omega))^{-1}\circ(-DG(\omega)) = (I+DG(\omega))^{-1}-I$, from (6.4) we deduce that

$$f''(\omega_s)\int_T L(t,s)K(s,t)\,dt$$

is $\mathcal{F}^i$-measurable, for $s\in D-(\partial D)_\varepsilon$.

We can apply again Lemma 2, and repeat the arguments above, now with a function $\psi\in\mathcal{C}^\infty_{\text{comp}}(D^c-(\partial D)_\varepsilon)$, to obtain

$$f''(\omega_s)f''(\omega_{\bar s})\big[K\cdot(I+DG(\omega))^{-1}\big](s,\bar s)\big[K\cdot(I+DG(\omega))^{-1}\big](\bar s,s)=0 \quad ,$$

for every $s\in D-(\partial D)_\epsilon$ and $\bar s\in D^c-(\partial D)_\epsilon$, from which it can be drawn that $f''(\omega_s)=0$. This implies obviously $f''=0$. Thus we get:
Theorem: If $f\in\mathcal{C}^2$, with $f'>0$, then (2.10) has a unique solution, which is a G.M.F. iff $f''=0$. □

7. Other methods and equations

The most serious objection that can be raised against the change of measure method is to have to deal with the Carleman–Fredholm determinant of a certain integral operator. In example 4 of the last Section its explicit computation has been avoided, but this is not always possible.

With this in mind, some different procedures have been developed recently. In Alabert and Nualart [2], a new approach was proposed and applied to the following second order difference equation:

$$\left.\begin{array}{c} X_{n+2}-2X_{n+1}+X_n = f(X_{n+1})+\xi_n \quad , \quad 0\le n\le N-2 \\ X_0=0,\ X_N=0 \end{array}\right\} \tag{7.1}$$

where the variables $\{\xi_n\}_{n=0}^{N-2}$ are given and assumed to be independent. In case ξ_n are absolutely continuous with a strictly positive density on $\mathbb{R}$, then the pair (X_n, X_{n+1}) is a Markov Process iff f is affine. But if ξ_n have discrete laws, then (X_n, X_{n+1}) is *always* a Markov Process.

The method used in [2] is based in an application of the co-area formula from Geometric Measure Theory, and leads in a natural way to raise the following general question: Given two independent random variables Z_1 and Z_2, are they conditionally independent given some function $g(Z_1, Z_2)$?

Ferrante and Nualart [10] applied more direct arguments to

$$\left.\begin{array}{c} X_{n+1} - X_n = f(X_n) + \sigma(X_n)\xi_n \quad , \quad 0 \le n \le N-1 \\ aX_0 + bX_N = c \end{array}\right\} \tag{7.2}$$

where f and σ are nonlinear and the noise is assumed positive and absolutely continuous. In this case, X_n is a M.F. iff

$$\left\{\begin{array}{l} x + f(x) = \beta x^\gamma \\ \sigma(x) = \alpha x^\gamma \end{array}\right.$$

for some $\alpha > 0$, $\beta > 0$ and $0 < \gamma \le 1$.

In Alabert, Ferrante and Nualart [1], the arguments used in these difference equations were refined and applied to

$$\left.\begin{array}{c} dX_t = f(t, X_{t-})\,\mu(dt) + dW_t \quad , \quad 0 \le t \le 1 \\ X_0 = \psi(X_1) \end{array}\right\} \tag{7.3}$$

where μ is any finite positive measure. The change of measure scheme does not work in this case. X_t is a M.F. iff one of following conditions holds:

a) $\psi' \equiv 0$.

b) $\forall t \in \operatorname{supp}\mu$, $f(t,\cdot)$ is affine.

c) ψ' is constant, and *b)* holds except possibly in one only point t_0.

It turns out that the investigation of the Markov Field property can be reduced to the following particular form of the general question cited above: Given two independent σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ of a probability space, and given two random variables determined as the solution of a system of the form

$$\left.\begin{array}{c} X = g_1(Y, \omega) \\ Y = g_2(X, \omega) \end{array}\right\}$$

where $g_i(y, \cdot)$ is $\mathcal{F}_i$-measurable ($i = 1, 2$), under what conditions on g_1 and g_2 are $\mathcal{F}_1$ and $\mathcal{F}_2$ conditionally independent given X and Y? The answer is given in [1], Theorem 2.1.

Finally, we remark that sometimes it is possible to make a change of variables to get rid of a nonlinear diffusion coefficient, thus making easier the investigation of conditional independence properties. For instance, assume b, σ and ψ are C^1 functions, with $\sigma > 0$. Then the problem

$$\left.\begin{array}{c} dX_t = b(X_t)\,dt + \sigma(X_t)\circ dW_t \quad , \quad 0 \le t \le 1 \\ X_0 = \psi(X_1) \end{array}\right\} \tag{7.4}$$

can be transformed in

$$\left.\begin{array}{c} dX_t = f(X_t)\,dt + dW_t \quad , \quad 0 \le t \le 1 \\ X_0 = \varphi(X_1) \end{array}\right\}$$

and one finds that the solution to (7.4) is a M.F. iff either

1) $\psi' \equiv 0$, or

2) $b(x) = A\sigma(x) + B\sigma(x)\int_c^x \frac{1}{\sigma(t)}\,dt$, for some constants A, B, c.

(see also [1] for a more detailed discussion).

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