On Numerical Approximation of Stochastic Burgers' Equation

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Summary. We present a finite difference scheme for stochastic Burgers' equation driven by space-time white noise. We estimate the rate of convergence of the the numerical scheme to the solution of stochastic Burgers's equation.

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1 Introduction

We consider stochastic Burgers' equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + u(t,x)\frac{\partial u}{\partial x}(t,x) + \frac{\partial W}{\partial t \partial x}(t,x), \tag{1.1}$$

for $t \in [0,T]$, $x \in [0,1]$, with Dirichlet boundary condition

$$u(t,0) = u(t,1) = 0, \quad t > 0, \tag{1.2}$$

and initial condition

$$u(0,x) = u_0(x)\ , \quad x \in [0,1]. \tag{1.3}$$

Here f is a Lipschitz continuous function on the real line, u_0 is a square-integrable function over $[0,1]$, and $\frac{\partial W}{\partial t \partial x}(t,x)$ is a space-time white noise. This equation is very often viewed as a model equation of the motion of turbulent fluid. The solvability and the properties of its solution have been intensively studied in the literature, see, e.g., [1], [2], [7] and the references therein. Our aim is to investigate a numerical scheme for this equation. We study the following space-discretization of problem (1.1)–(1.2):

$$\mathrm{d}u^n(t,x_k^n) = \Big(\Delta_n u^n(t,x_k^n) + f(u(t,x_k^n)) + \frac{1}{2}\partial_n^-[[u^n(t)]](x_k^n)\Big)\,\mathrm{d}t$$

$$+\mathrm{d}\partial_n W(t,x_k^n)\ , \quad k = 1,\dots,n-1, \tag{1.4}$$

$$u^n(t,x_0^n) = u^n(t,x_n^n) = 0, \quad t \geq 0, \tag{1.5}$$

over the grid $\mathcal{G}^n := \{x_k^n = k/n : \ k = 0,1,2,...,n\}$, where d stands for the differential in t, and

$$\Delta_n h(x_k^n) := n^2\Big(h(x_{k+1}^n) - 2h(x_k^n) + h(x_{k-1}^n)\Big),$$

$$\partial_n h(x_k^n) := n\Big(h(x_{k+1}^n) - h(x_k^n)\Big),$$

$$\partial_n^- h(x_k^n) := \Big(h(x_k^n) - h(x_{k-1}^n)\Big),$$

$$[[h]](x_k^n) := \frac{1}{3}\Big(h^2(x_{k+1}^n) + h^2(x_k^n) + h(x_{k+1}^n)h(x_k^n)\Big),$$

$$h(x_0^n) = h(x_n^n) := 0,$$

for functions h defined on the grid. For fixed $n \geq 2$ system (1.4) is a stochastic differential equation for the $(n-1)$-dimensional process

$$u^n(t) = (u_k^n)(t) := (u^n(t,x_k^n)).$$

We show that for every initial condition $u^n(0) = (a_k^n) \in \mathbb{R}^{n-1}$ equation (1.4) has a unique solution $\{u^n(t) : t \in [0,T]\}$. We extend $u^n(t)$ from the grid onto $[0,1]$ by $u^n(t,x) := u^n(t,[nx]/n)$, and show that this extension converges to u, the solution of stochastic Burgers' equation, provided that the initial condition $u^n(0)$ converges to u_0. Moreover, we estimate the rate of convergence.

Numerical schemes for parabolic stochastic PDEs driven by space-time white noise have been investigated thoroughly in the literature, see, e.g., [3], [6], [10], [11] and the references therein. The class of equations considered in these papers does not contain stochastic Burgers' equation. A semi-discretization in time of stochastic Burgers' equation is studied in [9].

2 Formulation of the main result

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leq t\leq T},P)$ be a filtered probability space carrying an $\mathcal{F}_t$-Brownian sheet $W = (W(t,x))$ on $[0,T]\times[0,1]$. This means W is a Gaussian field, $EW(t,x) = 0$, $E(W(t,x)W(s,y)) = (t \wedge s)(x \wedge y)$, $W(t,x)$ is $\mathcal{F}_t$-measurable, and $W(t,x) - W(s,x) + W(s,y) - W(t,y)$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t$ and $x, y \in [0,1]$.

Let $f := f(z)$ be a locally bounded Borel function on $\mathbb{R}$, and let $u_0 = u_0(x)$ be an $\mathcal{F}_0$-measurable random field such that almost surely $u_0 \in L^2([0,1])$. We say that an $L^2([0,1])$-valued continuous $\mathcal{F}_t$-adapted random process is a solution of problem (1.1), (1.2), (1.3), if almost surely

$$\int_0^1 u(t,x)\varphi(x)\,\mathrm{d}x = \int_0^1 u_0(x)\varphi(x)\,\mathrm{d}x + \int_0^t \int_0^1 u(s,x)\varphi''(x)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_0^1 f(u(s,x))\varphi(x)\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{2}\int_0^t \int_0^1 u^2(s,x)\varphi'(x)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_0^1 \varphi(x)\,\mathrm{d}W(s,x)$$

for all $t \in [0,T]$ and $\varphi \in C^2([0,1])$, $\varphi(0) = \varphi(1) = 0$, where the last integral in the right-hand side of this equality is understood as Itô's integral, and φ', φ'' denote the first and second derivatives of φ. We assume the following condition.

Assumption 2.1 *The force term f is Lipschitz continuous, i.e., there is a constant L such that*

$$|f(y) - f(z)| \le L|y - z|$$

for all $y, z \in \mathbb{R}$.

It is well-known that under this condition problem (1.1), (1.2), (1.3) has a unique solution u, which satisfies also the integral equation

$$u(t,x) = \int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G(t-s,x,y)f(u(s,y))\,\mathrm{d}y\,\mathrm{d}s$$
$$- \int_0^t \int_0^1 G_y(t-s,x,y)u^2(s,y)\,\mathrm{d}y\,\mathrm{d}s + \int_0^t \int_0^1 G(t-s,x,y)\,\mathrm{d}W(s,y), \quad (2.6)$$

where

$$G(t,x,y) := \sum_{j=1}^{\infty} \exp\{-j^2\pi^2 t\}\varphi_j(x)\varphi_j(y), \qquad \varphi_j(x) := \sqrt{2}\sin(j\pi x), \quad (2.7)$$

is the heat kernel, and

$$G_y(t,x,y) = \sum_{j=1}^{\infty} j\pi \exp\{-j^2\pi^2 t\}\varphi_j(x)\psi_j(y), \qquad \psi_j(x) := \sqrt{2}\cos(j\pi x). \quad (2.8)$$

Moreover, if u_0 is a continuous random field, then the solution u has a modification which is continuous in (t,x), see [1], [2] and [7].

First we formulate our result for problem (1.4)–(1.5).

Theorem 2.1. *Let Assumption 2.1 hold. Let $n \geq 2$ be an integer, and let $(a_k^n)_{k=1}^{n-1}$ be an $\mathcal{F}_0$-measurable random vector in $\mathbb{R}^{d-1}$. Then system (1.4)–(1.5) with the initial condition*

$$u^n(0, x_k^n) = a_k^n, \quad k = 1, 2, ..., n-1, \tag{2.9}$$

admits a unique solution $u^n = \{u^n(t, x_k^n) : k = 0, 1, 2, ..., n;\ t \geq 0\}$, which is continuous in $t \geq 0$. Moreover, for every $T > 0$, there is a finite random variable ξ such that

$$\sup_{t \leq T} \frac{1}{n} \sum_{j=1}^{n-1} |u^n(t, x_j^n)|^2 \leq \xi \left(\frac{1}{n} \sum_{j=1}^{n-1} |a_k^n|^2 + 1 \right) \quad (a.s.) \tag{2.10}$$

for all $n \geq 2$.

In order to formulate the main result of the paper we extend $(u^n(t, x_k^n))$, the solution of system (1.4)–(1.5) with initial condition $u^n(0, x_k^n) = u_0(x_k^n)$, $k = 0, 1, 2..., n$, as follows:

$$u^n(t, x) := u^n(t, \kappa_n(x)), \quad x \in [0, 1], \quad t \geq 0,$$

where $\kappa_n(x) := [nx]/n$, and $[z]$ denotes the integer part of z. The main result of the present paper is the following.

Theorem 2.2. *Let Assumption 2.1 hold. Assume that $u_0 \in C([0, 1])$ almost surely. Then $u^n(t)$ almost surely converges in $L_2([0, 1])$ to $u(t)$, the solution of problem (1.1)–(1.3), uniformly in t in bounded intervals. Moreover, if almost surely $u_0 \in C^3([0, 1])$, then for each $\alpha < 1/2$, $T > 0$ there exists a finite random variable ζ_α such that*

$$\sup_{t \leq T} \int_0^1 |u^n(t, x) - u(t, x)|^2 \, dx \leq \zeta_\alpha n^{-\alpha} \quad (a.s.) \tag{2.11}$$

for all integers $n \geq 2$.

We prove Theorem 2.1 in the next section, and after presenting some preliminary estimates in Section 4, we prove Theorem 2.2 in Section 5.

3 Proof of Theorem 2.1

Using the notation

$$u_k^n(t) := u^n(t, x_k^n) = u^n\Big(t, \frac{k}{n}\Big)$$

$$W_k^n(t) := \sqrt{n}\Big(W(t, x_{k+1}^n) - W(t, x_k^n)\Big)$$

for $k = 1, 2, \ldots, n-1$, we can write equations (1.4)–(1.5) as

$$\begin{aligned}
\mathrm{d}u_k^n(t) = n^2 \sum_{i=1}^{n-1} D_{ki} u_i^n(t)\,\mathrm{d}t + f(u_k^n(t))\,\mathrm{d}t \\
+\frac{n}{6}\Big(|u_{k+1}^n|^2(t) - |u_{k-1}^n|^2(t) + u_{k+1}^n(t)u_k^n(t) - u_k^n(t)u_{k-1}^n(t)\Big)\,\mathrm{d}t \\
+\sqrt{n}\,\mathrm{d}W_k^n(t), \quad k = 1,2,\dots,n-1, \qquad (3.12)\\
u_k^n(0) = a_k^n, \qquad k = 1,2,\dots,n-1, \qquad (3.13)
\end{aligned}$$

where $u_0^n = u_n^n := 0$, and $D_{kk} = -2$, $D_{ki} = 1$ for $|k-i| = 1$, $D_{ki} = 0$ for $|k-i| > 1$. Notice that $W^n(t) := (W_k^n(t))$ is an $(n-1)$-dimensional Wiener process. Fix $n \geq 2$ and define the vector field

$$A(x) := n^2 Dx + F(x) + nH(x), \quad x \in \mathbb{R}^{n-1},$$

where $D = (D_{ij})$ is the $(n-1)\times(n-1)$ matrix given above, and

$$\begin{aligned}
F_k(x_1, x_2, \dots, x_{n-1}) &: = f(x_k), \\
H_k(x_1, x_2, \dots, x_{n-1}) &: = \frac{1}{6}(x_{k+1}^2 - x_{k-1}^2 + x_{k+1}x_k - x_k x_{k-1}),
\end{aligned}$$

for $k = 1,2,\dots,n-1$, with $x_0 = x_n := 0$. Then equations (3.12)–(3.13) can be written as

$$\begin{aligned}
\mathrm{d}u^n(t) &= A(u^n(t))\,\mathrm{d}t + \sqrt{n}\,\mathrm{d}W^n(t), \qquad (3.14)\\
u^n(0) &= a^n, \qquad (3.15)
\end{aligned}$$

where $u^n(t) := (u_k^n(t))$ and $a^n := (a_k^n)$ are column vectors in $\mathbb{R}^{n-1}$. Notice that

$$\begin{aligned}
(x, Dx) &= -x_1^2 - x_{n-1}^2 - \sum_{k=1}^{n-2}(x_{k+1} - x_k)^2, \qquad (3.16)\\
(x, H(x)) &= 0, \qquad (3.17)\\
(x, F(x)) &= \sum_{k=1}^{n-1} x_k f(x_k) \leq C\left(n + \sum_{k=1}^{n-1} x_k^2\right) \qquad (3.18)
\end{aligned}$$

for all $x \in \mathbb{R}^{n-1}$, where $(x,y) := \sum_{k=1}^{n-1} x_k z_k$ is the inner product of vectors $x, y \in \mathbb{R}^{n-1}$, $C := L + f^2(0)$, and L is the Lipschitz constant from Assumption 2.1. Hence A satisfies the following growth condition:

$$(x, A(x)) = n^2(x, Dx) + (x, F(x)) \leq C\left(n + \sum_{k=1}^{n-1} x_k^2\right)$$

for all $x \in \mathbb{R}^{n-1}$ and for every integer $n \geq 2$. Clearly, A is locally Lipschitz in $x \in \mathbb{R}^{n-1}$. This and the above growth condition imply that equation (3.14) with initial condition (3.15) admits a unique solution u^n, which is an $\mathcal{F}_t$-adapted $\mathbb{R}^{n-1}$-valued continuous process. (See the general result, Theorem 1 in [4], or Theorem 3.1 in [8], for example.)

It remains to show estimate (2.10). To this end we rewrite equation (3.14) for the solution u^n in the form

$$\begin{aligned} u^n(t) = \mathrm{e}^{n^2tD}a^n + \int_0^t \mathrm{e}^{n^2(t-s)D}\Big(F(u^n(s)) + nH(u^n(s))\Big)\,\mathrm{d}s \\ +\sqrt{n}\int_0^t \mathrm{e}^{n^2(t-s)D}\,\mathrm{d}W^n(s), \end{aligned} \tag{3.19}$$

and consider the $\mathbb{R}^{n-1}$-valued random processes

$$\eta^n(t) := \sqrt{n}\int_0^t \mathrm{e}^{n^2(t-s)D}\,\mathrm{d}W^n(s), \quad v(t) := v^n(t) := u^n(t) - \eta^n(t).$$

Then from equation (3.19) we get that v satisfies

$$\begin{aligned} \mathrm{d}v(t) &= \Big(n^2Dv(t) + F(v(t)+\eta(t)) + nH(v(t)+\eta^n(t))\Big)\,\mathrm{d}t, \\ v(0) &= a^n. \end{aligned}$$

Hence for $|v(t)|^2 := \sum_{k=1}^{n-1}|v_k(t)|^2$ we get

$$\begin{aligned} \mathrm{d}|v(t)|^2 &= 2n^2\big(v(t), Dv(t)\big)\,\mathrm{d}t + 2\big(v(t), F(v(t)+\eta^n(t))\big)\,\mathrm{d}t \\ &\quad +2n\big(v(t), H(v(t)+\eta^n(t))\big)\,\mathrm{d}t \\ &\le -2n^2\sum_{k=1}^{n}(v_{k+1}(t)-v_k(t))^2\,\mathrm{d}t + 4C(n+|v(t)|^2) \\ &\quad +2n\big(v(t), H(v(t)+\eta^n(t)) - H(v(t))\big)\,\mathrm{d}t \end{aligned} \tag{3.20}$$

with $v_0(t) := v_n(t) := 0$, by virtue of (3.16), (3.17), (3.18), where C is the constant from inequality (3.18). Taking into account that for $x \in \mathbb{R}^{n-1}$

$$H_k(x) = [[x]]_k - [[x]]_{k-1}, \quad k = 1, \dots, n-1$$

with

$$[[x]]_j := \frac{1}{6}(x_{j+1}^2 + x_j^2 + x_{j+1}x_j), \quad j = 0, 1, \dots, n-1, \quad x_0 := x_n := 0,$$

we have

$$2|\big(v(t), H(v(t)+\eta(t)) - H(v(t))\big)| =$$

$$2|\sum_{k=0}^{n-1}(v_{k+1}(t)-v_k(t))\{[[v(t)+\eta^n(t)]]_k - [[v(t)]]_k\}|$$

$$\le n\sum_{k=0}^{n-1}(v_{k+1}(t)-v_k(t))^2+n^{-1}\sum_{k=0}^{n-1}\{[[v(t)+\eta^n(t)]]_k-[[v(t)]]_k\}^2$$

$$\le n\sum_{k=0}^{n-1}(v_{k+1}(t)-v_k(t))^2+100n^{-1}\sum_{k=1}^{n-1}\left(|\bar\eta_n|^2|v_k|^2(t)+|\bar\eta_n|^4\right), \tag{3.21}$$

where

$$\bar\eta^n := \max_{0<k<n}\sup_{t\le T}|\eta_k^n(t)|.$$

Thus from (3.20) and (3.21) we get

$$\frac{1}{n}|v(t)|^2\le\frac{1}{n}|v(0)|^2+100|\bar\eta^n|^4+4Ct+(100|\bar\eta^n|^2+4C)\int_0^t\frac{1}{n}|v(s)|^2\,\mathrm{d}s.$$

Hence by Gronwall's inequality

$$\sup_{t\le T}\frac{1}{n}|v(t)|^2\le \mathrm{e}^{(100|\bar\eta^n|^2+4C)T}\left(\frac{1}{n}|v(0)|^2+100|\bar\eta^n|^4+4CT\right),$$

which implies

$$\sup_{t\le T}\frac{1}{n}\sum_{k=1}^{n-1}|u_k^n(t)|^2\le\xi_n\left(\frac{1}{n}\sum_{k=1}^{n-1}|a_k^n|^2+1\right) \tag{3.22}$$

with

$$\xi_n := \mathrm{e}^{(100|\bar\eta^n|^2+4C)T}+100|\bar\eta^n|^4+4CT+2|\bar\eta^n|^2.$$

We are going to show that $\xi := \sup_{n\ge2}\xi_n$ is a finite random variable. To this end note that the vectors $e_1,\dots,e_{n-1}$ defined by

$$e_j=(e_j(k))=\left(\sqrt{\frac{2}{n}}\sin\left(j\frac{k}{n}\pi\right)\right),\qquad k=1,2,\dots,n-1,$$

form an orthonormal basis in $\mathbb{R}^{n-1}$, and that they are eigenvectors of the matrix n^2D, with eigenvalues

$$\lambda_j^n := -4\sin^2\left(\frac{j}{2n}\pi\right)n^2=-j^2\pi^2c_j^n,$$

where

$$\frac{4}{\pi^2}\le c_j^n := \sin^2\left(\frac{j\pi}{2n}\right)\Big/\left(\frac{j\pi}{2n}\right)^2\le 1 \tag{3.23}$$

for $j=1,2,\dots,n-1$ and every $n\ge1$. Therefore, for the random field $\{\eta^n(t,x): t\ge0, x\in[0,1]\}$ defined by

$$\eta^n(t,x_k) := \eta_k^n := \sqrt{n}\int_0^t \mathrm{e}^{n^2(t-s)D}\,\mathrm{d}W^n(s)$$

for $x_k := k/n$, $n = 1, 2, ..., n-1$, and

$$\eta^n(t,0) = \eta^n(t,1) = 0,$$

$$\eta^n(t,x) := \eta^n(t,\kappa_n(x)), \quad x \in (0,1),$$

we have

$$\eta^n(t,x) = \int_0^t \int_0^1 G^n(t,x,y)\,\mathrm{d}W(t,y),$$

for all $t \geq 0$, $x \in [0,1]$, where

$$G^n(t,x,y) := \sum_{j=1}^{n-1} \exp(\lambda_j^n t)\varphi_j^n(\kappa_n(x))\varphi_j(\kappa_n(y)), \tag{3.24}$$

$$\varphi_j(x) := \sqrt{2}\sin(jx\pi).$$

(Recall that $\kappa_n(y) := [ny]/n$.) Thus considering the special case $f = 0$, $\sigma = 1$, u_0 in Theorem 3.1 of [5], we get that almost surely

$$\sup_{n\geq 2} \bar{\eta}^n \leq \sup_{x\in[0,1]} \sup_{t\leq T} |\eta^n(t,x)| < \infty,$$

which obviously implies that $\xi := \sup_{n\geq 2} \xi_n$ is a finite random variable. The proof of Theorem 2.2 is now complete. □

4 Preliminary estimates

Define

$$G_y^n(t,x,y) := \partial_n G^n(t,x,y) := n(G^n(t,x,y+\frac{1}{n}) - G^n(t,x,y))$$

$$= \sum_{j=1}^{n-1} \exp\{-j^2\pi^2 c_j^n t\}\varphi_j(\kappa_n(x))n\big(\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y))\big), \tag{4.25}$$

for $t \geq 0$, $x, y \in [0,1]$, where $\kappa_n^+(y) =: \kappa_n(y) + \frac{1}{n}$.

Lemma 4.1. *For each $T > 0$ there exists a constant $K > 0$ such that*

$$\int_0^1 (G_y^n - G_y)^2(s,x,y)\,\mathrm{d}x = Kn^{-2}s^{-5/2}$$

for all $y \in [0,1]$, $s \in (0,T]$ and all integers $n \geq 2$.

Proof. Clearly,

$$G_y^n - G_y = A_1 + A_2 + A_3 + A_4 \ , \tag{4.26}$$

where

$$A_1 := \sum_{j=1}^{\infty} \exp\{-j^2\pi^2 s\}\big[\varphi_j(x) - \varphi_j(\kappa_n(x))\big] j\pi\psi_j(y) \ ,$$

$$A_2 := \sum_{j=n}^{\infty} \exp\{-j^2\pi^2 s\}\varphi_j(\kappa_n(x)) j\pi\psi_j(y) \ ,$$

$$A_3 := \sum_{j=1}^{n-1} \exp\{-j^2\pi^2 s\}\varphi_j(\kappa_n(x))\big[j\pi\psi_j(y) - n\big(\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y))\big)\big],$$

$$A_4 := \sum_{j=1}^{n-1} \ \{\big[\exp(-j^2\pi^2 s) - \exp(-j^2\pi^2 c_j^n s)\big] \\ \times\varphi_j(\kappa_n(x)) n\big(\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y))\big)\}.$$

Let $\|A_i\|$ denote the $L_2([0,1])$-norm of A_i in the x-variable. Fix $T > 0$, and let K denote constants, which are independent of $t \in [0,T]$, $x, y \in [0,1]$, $s \in (0,T]$, $n \geq 2$, but can be different even if they appear in the same line. Then notice that

$$\|A_1\|^2 = \int_0^1 \big|G_y(s,x,y) - G_y(s,x,y)\big|^2 \,\mathrm{d}x \\ \leq Kn^{-2} \int_0^1 \big|G_{yx}(s,x,y)\big|^2 \,\mathrm{d}x = Kn^{-2}s^{-5/2}, \tag{4.27}$$

by the well-known estimate

$$|G_{yx}(s,x,y)| \leq Ks^{-3/2}\mathrm{e}^{-(x-y)^2/s}, \quad s \in [0,T], \ \ x, y \in [0,1],$$

on the heat kernel. By the orthogonality of $\{\varphi_j\}$ in $L_2([0,1])$,

$$\|A_2\|^2 = \sum_{j=n}^{\infty} \exp\{-2j^2\pi^2 s\} j^2\pi^2\psi_j(y)^2 \\ \leq \sum_{j=n}^{\infty} j^2 \exp\{-j^2 s\} \leq 32 \sum_{j=n}^{\infty} j^2 \frac{1}{(js^{1/2})^5} \leq Kn^{-2}s^{-5/2}. \tag{4.28}$$

By the mean-value theorem

$$\|A_3\|^2 = \sum_{j=1}^{n-1} \exp\{-2j^2\pi^2 s\}\big[j\pi\psi_j(y) - n\big(\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y))\big)\big]^2$$

$$= \sum_{j=1}^{n-1} \exp\{-2j^2\pi^2 s\}\big[j\pi\psi_j(y) - j\pi\psi_j(\theta_n(y))\big]^2 \ ,$$

where $\theta_n(y) \in [\kappa_n(y), \kappa_n^+(y)]$. Hence

$$\begin{aligned}\|A_3\|^2 &\le Kn^{-2} \sum_{j=1}^{n-1} j^4 \exp\{-j^2 s\} \le Kn^{-2}s^{-2} \sum_{j=1}^{n-1} j^4 s^2 \exp\{-j^2 s\} \\ &\le Kn^{-2}s^{-2} \int_0^{n\sqrt{s}} x^4 \exp\{-x^2\} s^{-1/2}\,\mathrm{d}x \le Kn^{-2}s^{-5/2}. \qquad (4.29)\end{aligned}$$

Finally,

$$\begin{aligned}\|A_4\|^2 &= \sum_{j=1}^{n-1} \big[\exp\{-j^2\pi^2 s\} - \exp\{-j^2\pi^2 c_j^n s\}\big]^2 n^2 \big[\varphi_j(\kappa_n^+(y)) - \varphi_j(\kappa_n(y))\big]^2 \\ &\le K \sum_{j=1}^{n-1} j^2 \big[\exp\{-j^2\pi^2 s\} - \exp\{-j^2\pi^2 c_j^n s\}\big]^2 \\ &\le K \sum_{j=1}^{n-1} j^2 \big[j^2\pi^2 \exp\{-j^2\pi^2 c_j^n s\}(1-c_j^n)s\big]^2 \\ &\le K \sum_{j=1}^{n-1} j^6 (1-c_j^n)^2 s^2 \exp\{-j^2 s\}\end{aligned}$$

by the mean-value theorem and the fact that $c_j^n \le 1$. Hence by the definition of c_j^n in (3.23), using $\sin x = x + O(x^3)$, we have

$$\begin{aligned}\|A_4\|^2 &\le K \sum_{j=1}^{n-1} j^6 (j\pi/2n)^4 s^2 \exp\{-j^2 s\} \le K \sum_{j=1}^{n-1} j^6 (j/n)^4 s^2 \exp\{-j^2 s\} \\ &\le Kn^{-4} \sum_{j=1}^{n-1} j^{10} s^2 \exp\{-j^2 s\} \le Kn^{-2}s^{-2} \sum_{j=1}^{n-1} j^8 s^4 \exp\{-j^2 s\} \\ &\le Kn^{-2}s^{-2} \int_0^{s\sqrt{n}} x^8 \exp\{-x^2\} s^{-1/2}\,\mathrm{d}x \le Kn^{-2}s^{-5/2}\ . \qquad (4.30)\end{aligned}$$

Thus by virtue of equality (4.26) and inequalities (4.27), (4.28), (4.29) and (4.30) the proof is complete. □

Lemma 4.2. *For each $T > 0$ there exists a constant K such that*

$$I := \int_0^T \Big(\int_0^1 |G_y^n - G_y|^2(s,x,y)\,\mathrm{d}x\Big)^{1/2} \mathrm{d}s \le Kn^{-1/2} \qquad (4.31)$$

for all $y \in [0,1]$.

Proof. Clearly, $I \le I_1 + I_2 + I_3$, where

$$I_1 := \int_0^\varepsilon \Big(\int_0^1 G_y(s,x,y)^2 \,\mathrm{d}x \Big)^{1/2} \mathrm{d}s,$$

$$I_2 := \int_0^\varepsilon \Big(\int_0^1 G_y^n(s,x,y)^2 \,dx \Big)^{1/2} \mathrm{d}s\,\mathrm{d}y,$$

$$I_3 := \int_\varepsilon^T \int_0^1 (G_y^n - G_y)^2(s,x,y)\,\mathrm{d}x \Big)^{1/2} \mathrm{d}s\,\mathrm{d}y.$$

From

$$G_y(s,x,y) = \sum_{j=1}^\infty \exp(-j^2\pi^2 s)\varphi_j(x) j\pi\psi_j(y),$$

using the orthogonality of $\{\varphi_j\}$, we get

$$\begin{aligned}\int_0^1 G_y(s,x,y)^2 \,\mathrm{d}x &\le \sum_{j=1}^\infty \exp(-2j^2\pi^2 s) j^2\pi^2 \psi_j^2(y) \\ &\le 20 \sum_{j=1}^\infty \exp(-j^2 s) j^2 \le C s^{-3/2}\end{aligned}$$

for some constant C. Therefore,

$$I_1 \le \int_0^\varepsilon C s^{-3/4} \,\mathrm{d}s \le 4C\varepsilon^{1/4}.$$

In exactly the same way, we obtain a constant C such that $I_2 \le C\varepsilon^{1/4}$. By the estimate in Lemma 4.1, there is a constant C such that

$$I_3 \le Cn^{-1} \int_\varepsilon^T s^{-5/4} \,\mathrm{d}s\,\mathrm{d}y \le Cn^{-1}\varepsilon^{-1/4}.$$

Taking $\varepsilon = n^{-2}$, we obtain the statement of the lemma. □

5 Proof of Theorem 2.2

We prove the theorem when $f = 0$. The proof in the general case of a Lipschitz function f goes in the same way, with some additional terms in the calculations, but without new difficulties. Notice that $u^n(t,x)$ satisfies

$$\begin{aligned}u^n(t,x) = \quad & \int_0^1 G^n(t,x,y) u(0,\kappa_n(y))\,\mathrm{d}y \\ & - \int_0^t \int_0^1 G_y^n(t-s,x,y)[[u^n(s)]](\kappa_n(y))\mathrm{d}y\,\mathrm{d}s \\ & + \int_0^t \int_0^1 G^n(t-s,x,y)\,\mathrm{d}W(s,y), \end{aligned} \tag{5.32}$$

where G^n and G^n_y are defined by (4.25) and (4.25), respectively. From equations (2.6) and (5.32)

$$\|u^n(t,\cdot)-u(t,\cdot)\| \le A(t)+B(t)+C(t), \tag{5.33}$$

with

$$A(t) := \| \int_0^1 G^n(t,\cdot,y)u_0^n(y)\,\mathrm{d}y - \int_0^1 G(t,\cdot,y)u_0(y)\,\mathrm{d}y\|, \tag{5.34}$$

$$\begin{aligned} B(t) := \| &\int_0^t \int_0^1 G_y(t-s,\cdot,y)u(s,y)^2\,\mathrm{d}y\,\mathrm{d}s \\ &- \int_0^t \int_0^1 G^n_y(t-s,\cdot,y)[[u^n(s)]](\kappa_n(y))\,\mathrm{d}y\,\mathrm{d}s\|, \end{aligned}$$

$$C(t) := \| \int_0^t \int_0^1 G^n(t-s,x,y)\,\mathrm{d}W(s,y) - \int_0^t \int_0^1 G(t-s,x,y)\,\mathrm{d}W(s,y)\|. \tag{5.35}$$

Clearly, $B \le B_1 + B_2$, where

$$B_1^2(t) := \int_0^1 \Big(\int_0^t \int_0^1 (G^n_y - G_y)(t-s,x,y)[[u^n(s)]](y)\,\mathrm{d}y\,\mathrm{d}s \Big)^2 \mathrm{d}x,$$

$$B_2^2(t) := \int_0^1 \Big(\int_0^t \int_0^1 G_y(t-s,x,y)([[u^n(s)]](y) - |u(s,y)|^2)\,\mathrm{d}y\,\mathrm{d}s \Big)^2 \mathrm{d}x.$$

By Minkowski's inequality, Lemma 4.2 and Theorem 2.1 we get

$$\begin{aligned} B_1^2(t) &\le \Big(\int_0^1 \int_0^t \Big(\int_0^1 (G^n_y - G_y)^2(s,x,y)\,\mathrm{d}x \Big)^{1/2} [[u^n(t-s)]](y)\,\mathrm{d}s\,\mathrm{d}y \Big)^2 \\ &\le Kn^{-1} \Big(\int_0^t \int_0^1 [[u^n(s)]](y)\,\mathrm{d}y\,\mathrm{d}s \Big)^2 \le \xi n^{-1} \end{aligned} \tag{5.36}$$

for all $t \in [0,T]$, where K is a constant and ξ is a finite random variable, independent of t and n. By Lemma 3.1 (i) from [7], (take $q=1$, $\rho=2$, $\kappa=1/2$ there), we have

$$B_2^2(t) \le K \Big(\int_0^t (t-s)^{-3/4} \|[[u^n(s,\cdot)]] - |u(s,\cdot)|^2\|_1\,\mathrm{d}s \Big)^2 \tag{5.37}$$

for all $t \in [0,T]$, where $\|\cdot\|_1$ denotes the $L_1([0,1])$-norm. By simple calculations, using the Cauchy–Bunyakovskii inequality we get

$$\begin{aligned} \|[[u^n(s,\cdot)]] - |u(s,\cdot)|^2\|_1 \le K&\|u^n(s,\cdot)-u(s,\cdot)\|(\|u^n(s,\cdot)\| + \|u(s,\cdot\|) \\ &+K\|u(s,\cdot)-u(s,\cdot+n^{-1})\|\|u^n(s,\cdot\| \end{aligned} \tag{5.38}$$

for all $s \in [0,T]$ with a constant K. By Theorem 2.1 and Theorem 1 in [7], there is a finite random variable ξ such that almost surely

$$\|u^n(s,\cdot)\|^2 \le \xi, \quad \|u(s,\cdot)|^2 \le \xi$$

for all $s \in [0,T]$ and integers $n \ge 2$. Thus from (5.38) and (5.37) by Jensen's inequality we obtain

$$|B_2(t)|^2 \le \xi \int_0^t (t-s)^{-3/4} \|u^n(s,\cdot) - u(s,\cdot)\|^2 \,\mathrm{d}s + \xi\zeta_n \tag{5.39}$$

for all $t \in [0,T]$ and $n \ge 2$, where

$$\zeta_n := \sup_{s \le T} \|u(s,\cdot) - u(s,\cdot + n^{-1})\|^2, \tag{5.40}$$

and ξ is a finite random variable independent of t and n. By Burkholder's inequality for every $p \ge 1$ there exists a constant K_p such that

$$E\left[\sup_{t \le T} |C(t)|^{2p}\right] \le K_p \Big\| \int_0^t \int_0^1 (G^n - G)^2(t-s,\cdot,y) \,\mathrm{d}y \,\mathrm{d}s \Big\|_p,$$

where $\|\cdot\|_p$ stands for the $L_p([0,1])$ norm. Consequently, for each $p \ge 1$ there exists a constant C_p such that

$$E\left[\sup_{t \le T} |C(t)|^{2p}\right] \le C_p n^{-p},$$

since

$$\sup_{x \in [0,1]} \int_0^\infty \int_0^1 |G^n - G|^2(t,x,y) \,\mathrm{d}y \,\mathrm{d}t \le \frac{c}{n}$$

with a universal constant c by Lemma 3.2 part (i) in [5]. Hence, by standard arguments, for any $\alpha \in (0,1)$, one gets a finite random variable ξ_α such that almost surely

$$\sup_{t \le T} |C(t)|^2 \le \xi_\alpha n^{-\alpha} \tag{5.41}$$

for all $n \ge 2$. From (5.33) (5.36), (5.39) and (5.41) we get that almost surely

$$\begin{aligned} \|u^n(t,\cdot) - u(t,\cdot)\|^2 \le \xi \int_0^t (t-s)^{-3/4} \|u^n(s,\cdot) - u(s,\cdot)\|^2 \,\mathrm{d}s \\ +\xi(\zeta_n + |A(t)|^2 + n^{-1}) + \xi_\alpha n^{-\alpha} \end{aligned}$$

for all $t \in [0,T]$, and integers $n \ge 2$, with a finite random variable ξ, where $A(t)$, ζ_n and ξ_α are defined in (5.34), (5.40) and (5.41), respectively. Hence, applying a Gronwall-type lemma (e.g. Lemma 3.4 from [5]), we obtain that almost surely

$$\sup_{t\leq T}\|u^n(t,\cdot)-u(t,\cdot)\|^2 \leq \xi\left(\zeta_n+\sup_{t\leq T}|A(t)|^2+n^{-1}+\xi_\alpha n^{-\alpha}\right) \tag{5.42}$$

Now we are going to investigate the behaviour of $A(t)$ and ζ_n as $n\to\infty$. Set

$$v^n(t,x) := \int_0^1 G^n(t,x,y)u_0(\kappa_n(y))\,\mathrm{d}y$$
$$v(t,x) := \int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y.$$

Assume that $u_0\in C^3([0,1])$. Then by Proposition 3.8 in [5] we have a finite random variable ξ such that almost surely

$$\sup_{t,\in[0,T]}\sup_{x\in[0,1]}|v^n(t,x)-v(t,x)|\leq \xi n^{-1}$$

for all $n\geq 2$. Hence almost surely

$$\sup_{t\in[0,T}|A(t)|^2=\int_0^1|v^n(t,x)-v(t,x)|^2\,\mathrm{d}x\leq \xi^2 n^{-2} \tag{5.43}$$

for all $t\in[0,T]$ and integers $n\geq 2$. Moreover, using Lemma 3.1 (iii) from [7] (with $\rho=2$, $q=1$ and $\kappa=1/2$ there), we get a finite random variable ξ, such that almost surely

$$\zeta_n := \sup_{s\leq T}\|u(s,\cdot)-u(s,\cdot+n^{-1})\|^2\leq \xi n^{-1} \tag{5.44}$$

for all $n\geq 2$. Consequently, inequalities (5.42), (5.43) and (5.44) imply estimate (2.11) of Theorem 2.2. Assume now that $u_0\in C([0,1])$. Then by Lemma 3.1 (iii) from [7] and Proposition 3.8 in [5] we have that almost surely

$$\sup_{t\in[0,T]}A(t)+\zeta_n\to 0,$$

as $n\to\infty$. Hence as $n\to\infty$,

$$\sup_{t\leq T}\|u^n(t,\cdot)-u(t,\cdot)\|^2\to 0\quad (a.s.).$$

The proof of Theorem 2.2 is complete. □

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