A Conditional Independence Property for the Solution of a Linear Stochastic Differential Equation with Lateral Conditions

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Abstract. Let L be an nth order linear differential operator with smooth coefficients and $\{W(t) : t \in [0,1]\}$ a standard Wiener process. We consider the stochastic differential equation

$$L[X] = \dot{W}$$

on $[0,1]$, with the lateral condition

$$\sum_{j=1}^{m} \alpha_{ij} X(t_j) = c_i \quad , \quad 1 \leq i \leq n \quad ,$$

where $0 \leq t_1 < \ldots < t_m \leq 1$ and $\alpha_{ij}, c_i \in \mathbb{R}$. We prove that the solution to this system, considered as the vector $Y(t) = \left(X^{(n-1)}(t), \ldots, X'(t), X(t)\right)$, is not a Markov field in general but satisfies a weaker conditional independence property.

1 Introduction

In the last few years there has been some work on stochastic differential equations (SDEs) with boundary conditions (see e.g. [8], [6], [7], [1], [2]). This means, SDE driven by white noise on a compact time interval, say $[0,1]$, where instead of the customary initial condition, a relationship $h(X_0, X_1) = 0$ is imposed between the first and the last variable of the solution process.

Due to this relationship, the existence of a solution adapted to the driving process cannot be expected and in some instances the theory and techniques of the recently developed anticipating stochastic calculus have to be employed.

Besides the fundamental problem of existence and uniqueness, the main interest has resided in the study of some suitable Markov-type property for the solution process. In most cases, the boundary condition prevents the Markov process property from holding, and weaker conditional independence properties (e.g. the Markov field property, see Definition 3.1) have been considered.

The case of first order equations with linear coefficients and linear boundary condition was studied at length by Ocone and Pardoux [8] (see also [4]). Most of the subsequent papers centered on obtaining necessary and sufficient conditions on the coefficients of some nonlinear equation for the solution to be a Markov field or to enjoy a similar property, for some specific boundary condition.

In the present paper we fix a linear SDE and study a conditional independence property for its solution when subject to more general conditions. Specifically, we consider a linear SDE of arbitrary order with additive white noise on the right-hand side and additional linear conditions which may involve the value of the solution process at some points in the interior of the time interval.

2 Linear SDE with lateral conditions

Let $I = [0,1]$ and consider the differential operator

$$L = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 \quad , \quad D = \frac{d}{dt} \quad ,$$

where $a_i \in C^\infty(I)$, $0 \leq i \leq n-1$. Let $\{W(t) : t \in I\}$ be a standard Wiener process. We assume that W is the coordinate process in the classical Wiener space $(C(I), \mathcal{B}(C(I)), P)$. We shall consider the SDE

$$L\,[X] = \dot{W} \tag{2.1}$$

together with the lateral condition

$$\sum_{j=1}^{m} \alpha_{ij} X(t_j) = c_i \quad , \quad 1 \leq i \leq n \quad , \tag{2.2}$$

where $n \leq m$, $0 \leq t_1 < \ldots < t_m \leq 1$, and α_{ij}, c_i are real numbers. The matrix of coefficients (α_{ij}) is assumed to have full rank.

As in the case of ordinary differential equations, (2.1) can be regarded as a first order system

$$DY(t) + A(t)\,Y(t) = \dot{B}(t) \quad , \quad t \in I \quad , \tag{2.3}$$

where $Y(t) = (Y_1(t), \ldots, Y_n(t))$, $Y_i(t) = D^{n-i}X(t)$ for $1 \leq i \leq n$, $B(t) = (W(t), 0, \ldots, 0)$, and

$$A(t) = \begin{bmatrix} a_{n-1}(t) & a_{n-2}(t) & a_{n-3}(t) & \cdots & a_1(t) & a_0(t) \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{bmatrix} . \tag{2.4}$$

The lateral condition (2.2) is a special case of the general linear condition

$$\Lambda\,[X] \;=\; c \quad , \tag{2.5}$$

with Λ a linear $\mathbb{R}^n$-valued functional on $C^{n-1}(I)$, and $c \in \mathbb{R}^n$. By the Riesz representation theorem, (2.5) can be written as

$$\int_0^1 dF(t)\; Y(t) \;=\; c \quad , \tag{2.6}$$

where F is an $(n \times n)$-matrix whose components are functions of bounded variation on I.

When the right-hand side of (2.3) is a continuous vector function g, it is well known that the system

$$\begin{cases} DY(t) \;+\; A(t)\; Y(t) \;=\; g(t) \quad , \quad t \in I \\[2ex] \displaystyle\int_0^1 dF(t)\; Y(t) \;=\; c \end{cases} \tag{2.7}$$

admits a unique solution, which belongs to $C^1(I; \mathbb{R}^n)$, if and only if for some $s \in I$ (equivalently, for every $s \in I$)

(H) $$\det \int_0^1 dF(t)\; \Phi^s(t) \;\neq\; 0 \quad ,$$

where $\Phi^s(t)$ denotes the fundamental matrix solution of $DY(t) + A(t)Y(t) = 0$, that is, $\forall s \in I$,

$$\begin{cases} \dfrac{d}{dt}\Phi^s(t) + A(t)\; \Phi^s(t) = 0 \quad , \quad t \in I \\[2ex] \Phi^s(s) = \mathrm{Id} \quad , \end{cases}$$

with Id the identity matrix. When **(H)** holds, the solution to (2.7) is given by

$$Y(t) \;=\; J(t)^{-1}\; c \;+\; \int_0^1 G(t,s)\; g(s)\; ds \quad ,$$

where

$$J(t) = \int_0^1 dF(u)\; \Phi^t(u) \tag{2.8}$$

and $G(t,s)$ is the (matrix-valued) Green function associated to A and F. An explicit expression for this function is the following (see e.g. [3] or [5]):

$$G(t,s) \;=\; J(t)^{-1} \left[\int_0^s dF(u)\; J(u)^{-1} \;-\; \mathbf{1}_{\{t \leq s\}}\; \mathrm{Id} \right] J(s) \quad . \tag{2.9}$$

Under **(H)**, we define the solution to (2.3)-(2.2) as the n-dimensional stochastic process

$$Y(t) \;=\; J(t)^{-1}\, c \;+\; \int_0^1 G(t,s)\; dB(s) \quad , \tag{2.10}$$

and the solution to (2.1)-(2.2) as the process $\{X(t) = Y_n(t),\ t \in I\}$. The Green function (2.9) has bounded variation, so that the Wiener integrals in (2.10) can be interpreted pathwise by means of an integration by parts

$$\Big[\int_0^1 G(t,s)\, dB(s)\Big](\omega) = -\int_0^1 G(t,ds)\, B(s)(\omega)$$

(we take into account that $G(t,1) = 0,\ \forall t$), and therefore Y can be defined everywhere. We shall assume throughout the paper that the solution is taken in this pathwise sense. Furthermore, it is not difficult to show that the process $Y(t)$ so defined is continuous (hence $X(t)$ is a C^{n-1} process) and that, for each $t \in I$, the mapping $\omega \mapsto Y(t)(\omega)$ is continuous from $C(I)$ into $\mathbb{R}^n$.

Notice that, with the notation introduced in (2.6), the particular lateral condition (2.2) corresponds to

$$dF \;=\; \begin{bmatrix} 0 & 0 & \cdots & 0 & \sum_{j=1}^m \alpha_{1j}\delta_{t_j} \\ 0 & 0 & \cdots & 0 & \sum_{j=1}^m \alpha_{2j}\delta_{t_j} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & \sum_{j=1}^m \alpha_{nj}\delta_{t_j} \end{bmatrix} \quad , \tag{2.11}$$

where δ_t denotes Dirac measure at t, and $J_{ik}(t) = \sum_{j=1}^m \alpha_{ij}\Phi^t_{nk}(t_j)$. Notice also that only the first column of $G(t,s)$ is relevant in (2.10).

Example: Nualart and Pardoux [7] studied the following second order SDE with boundary conditions:

$$\begin{cases} D^2X(t) + f\big(DX(t), X(t)\big) = \dot{W}(t) \quad , \quad t \in I \\ X(0) = c_1 \quad , \quad X(1) = c_2 \quad . \end{cases} \tag{2.12}$$

In Proposition 1.5 of [7] the authors prove directly that in case f is an affine function $f(x,y) = a_1x + a_0y + b$, (2.12) admits a unique solution if

$$\int_0^1 \Big(\exp\big[(s-1)A\big]\Big)_{21}\ (a_0s + a_1)\ ds \neq 1 \quad , \tag{2.13}$$

where the subindex 21 means taking the entry of second row and first column.

This condition is in fact a particular case of **(H)** and therefore is also necessary. Indeed, with the formalism of (2.6), we have that the boundary condition corresponds to

$$dF \;=\; \begin{bmatrix} 0 & \delta_0 \\ 0 & \delta_1 \end{bmatrix} \quad .$$

Recall that if the matrices $A(t)$ and $\int_0^t A(s)\, ds$ commute for every t, then the fundamental solution $\Phi^0(t)$ is given by $\exp\{-\int_0^t A(s)\, ds\}$. This is the case when $A(t)$ is constant. By **(H)**, we have that (2.12) admits a unique solution if and only if $(\exp[-A])_{21} \neq 0$. Noticing that $D\,(\exp[-At])_{21} = (\exp[-At])_{11}$ and that $D\,(\exp[-At])_{11} = -\,\alpha_1\,(\exp[-At])_{11} - \,\alpha_0\,(\exp[-At])_{21}$, it is easy to prove that the integral in (2.13) is equal to $1 - (\exp[-A])_{21}$.

3 A Markov-type property

In the study of boundary value stochastic problems, the authors have examined which conditions on the coefficients of the equation make the solution process satisfy some suitably defined Markov-type property. Intuition suggests that a relation $h(X_0, X_1) = 0$ will possibly prevent the Markov process property from holding in general. One might think that nevertheless the Markov field property, which can be defined as follows, will be satisfied. It is easy to see that any Markov process is a Markov field. The converse is not true in general.

Definition 3.1. *A continuous process* $\{X_t, t \in I\}$ *is said to be a* Markov field *if for any* $0 \leq a < b \leq 1$ *, the* σ*-fields* $\sigma\{X_t, t \in [a,b]\}$ *and* $\sigma\{X_t, t \in [0,a] \cup [b,1]\}$ *are conditionally independent given* $\sigma\{X_a, X_b\}$.

However, even this weaker property holds only in special cases. For instance, in [1] it was shown that the solution to

$$\begin{cases} DX(t) = b(X(t)) + \sigma(X(t)) \circ \dot{W}(t) \quad , \quad t \in I \\ \\ X(0) = \psi(X(1)) \end{cases} \tag{3.1}$$

is a Markov field if and only if (essentially) $b(x) = A\sigma(x) + B\sigma(x) \displaystyle\int_c^x \frac{1}{\sigma(t)}\, dt$, for some constants A, B, c. As a corollary, in case σ is a constant (additive noise), X is a Markov field if and only if b is an affine function.

What can we say in such linear–additive cases when the additional condition takes into account the value of the solution in some interior points of the time interval? The following simple example illustrates how the situation may change.

Example: Consider the first order SDE with a lateral condition

$$\begin{cases} DX(t) \;=\; \dot{W}(t) \quad , \quad t \in I \\ \\ X(\frac{1}{2}) + X(1) \;=\; 0 \quad . \end{cases} \tag{3.2}$$

The solution is the process

$$X(t) = -\tfrac{1}{2}\big(W(\tfrac{1}{2}) + W(1)\big) + W(t) \quad ,$$

which is not a Markov field. Indeed, for $a = 0$ and $b = \frac{2}{3}$, the random variables $X(\frac{1}{2})$ and $X(1)$ are not conditionally independent given $\sigma\{X(a), X(b)\}$. Nevertheless, X is a Markov field when restricted to $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$.

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We will write in the sequel

$$\mathcal{F}_1 \underset{\mathcal{G}}{\perp\!\!\!\perp} \mathcal{F}_2$$

to mean that the σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given the σ-field $\mathcal{G}$.

In the present section we are going to prove for our equation (2.3)-(2.2) the following property:

(P) If $[a, b] \cap \{t_1, \dots, t_m\} = \emptyset$, then

$$\sigma\{Y(t), t \in [a, b]\} \underset{\sigma\{Y(a),Y(b)\}}{\perp\!\!\!\perp} \sigma\{Y(t), t \in]a, b[^c\} \quad , \tag{3.3}$$

except maybe for some singular pairs $(a, b) \in I^2$, $a \leq b$ (see Assumption **(A)** below). It is easily seen that **(P)** is satisfied by the solution to (3.2), for all $a, b \in I$.

To prove that the solution of (2.3)-(2.2) satisfies property **(P)**, we shall use a multidimensional version of Theorem 2.1 in [1] on the characterization of conditional independence in terms of a factorization property. This version was stated in [4], and we recall it here.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1$ and $\mathcal{F}_2$ two independent sub-σ-fields of $\mathcal{F}$. Consider two functions $g_1 : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ and $g_2 : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ such that g_i is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_i$–measurable, $i = 1, 2$. Set $B(\varepsilon) := \{x \in \mathbb{R}^d, |x| < \varepsilon\}$, and denote by λ the Lebesgue measure on $\mathbb{R}^d$. Let us introduce the following hypotheses:

(H1) There exists $\varepsilon_0 > 0$ such that for almost all $\omega \in \Omega$, and for any $|\xi| < \varepsilon_0$, $|\eta| < \varepsilon_0$ the system

$$\begin{cases} x - g_1(y, \omega) = \xi \\ y - g_2(x, \omega) = \eta \end{cases} \tag{3.4}$$

has a unique solution $(x, y) \in \mathbb{R}^{2d}$.

(H2) For every $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, the random vectors $g_1(y, \cdot)$ and $g_2(x, \cdot)$ possess absolutely continuous distributions and the function

$$\delta(x, y) = \sup_{0<\varepsilon<\varepsilon_0} \frac{1}{[\lambda(B(\varepsilon))]^2} P\{|x - g_1(y)| < \varepsilon, |y - g_2(x)| < \varepsilon\}$$

is locally integrable in $\mathbb{R}^{2d}$, for some $\varepsilon_0 > 0$.

(H3) For almost all $\omega \in \Omega$, the functions $y \mapsto g_1(y,\omega)$ and $x \mapsto g_2(x,\omega)$ are continuously differentiable and

$$\sup_{\substack{|y-g_2(x,\omega)|<\varepsilon_0 \\ |x-g_1(y,\omega)|<\varepsilon_0}} \left|\det\left[\mathrm{Id} - \nabla g_1(y,\omega)\ \nabla g_2(x,\omega)\right]\right|^{-1} \in L^1(\Omega)$$

for some $\varepsilon_0 > 0$, where ∇g_i denotes the Jacobian matrix of g_i with respect to the first argument.

Notice that hypothesis **(H1)** implies the existence of two random vectors X and Y determined by the system

$$\begin{cases} X(\omega) = g_1(Y(\omega),\omega) \\ \\ Y(\omega) = g_2(X(\omega),\omega) \end{cases} \tag{3.5}$$

Theorem 3.1. *Suppose the functions g_1 and g_2 satisfy the above hypotheses (H1) to (H3). Then the following statements are equivalent:*

(i) *$\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given the random vectors X, Y.*

(ii) *There exist two functions $F_i : I\!R^{2d} \times \Omega \to I\!R$, $i = 1, 2$, which are $\mathcal{B}(I\!R^{2d}) \otimes \mathcal{F}_i$-measurable, such that*

$$\left|\det\left[\mathrm{Id} - \nabla g_1(Y)\ \nabla g_2(X)\right]\right| = F_1(X,Y,\omega)F_2(X,Y,\omega) \quad , \quad a.s.$$

To apply Theorem 3.1 in proving a conditional independence with respect to $\sigma\{Y(a), Y(b)\}$, we will split $(Y(a), Y(b))$ into two vectors T^1 and T^2 of the same dimension (vectors X, Y in Theorem 3.1) in such a way that T^2 is determined by T^1 and the increments of the Wiener process W in $[a, b]$, and in turn T^1 is determined by T^2 and the increments of W in $]a, b[^c$ (functions g_1 and g_2 above).

In general it is not true that the lateral value problems that should define g_1 and g_2 are well-posed. We will plainly exclude those points a, b which fail to have a unique solution. Specifically:

(A) Suppose $0 \le t_1 < \ldots < t_k < a < b < t_{k+1} < \ldots < t_m \le 1$ and let ℓ and p be the number of lateral conditions that do not involve points in $[b, 1]$ and $[0, a]$, respectively, i.e.

$$\begin{aligned} \ell &= \#\{i\ : t_j > b \Rightarrow \alpha_{ij} = 0\} \quad , \\ p &= \#\{i\ : t_j < a \Rightarrow \alpha_{ij} = 0\} \quad . \end{aligned} \tag{3.6}$$

Consider the following sets of lateral conditions:

a)

$$\left\{\begin{array}{lll} Y_i(b) = 0 & , & i = 1,\dots,n-\ell \\ Y_i(a) = 0 & , & i = n-\ell+1,\dots,n \end{array}\right. \quad , \quad \text{on } [a,b] \ ;$$

b)

$$\left\{\begin{array}{lll} Y_i(a) = 0 & , & i = 1,\dots,n-\ell \\ \displaystyle\sum_{j=1}^{k} \alpha_{ij}\, Y_n(t_j) = 0 & , & i = 1,\dots,\ell \ . \end{array}\right. \quad , \quad \text{on } [0,a] \ ;$$

c)

$$\left\{\begin{array}{lll} Y_i(b) = 0 & , & i = n-\ell+1,\dots,n \\ \displaystyle\sum_{j=k+1}^{m} \alpha_{ij}\, Y_n(t_j) = 0 & , & i = n-p+1,\dots,n \\ \displaystyle\sum_{j=1}^{m} \alpha_{ij}\, Y_n(t_j) = 0 & , & i = \ell+1,\dots,n-p \\ Y_n(t_j) = 0 & , & j = 1,\dots,k \end{array}\right. \quad , \quad \text{on } [b,1] \ ,$$

(notice that the third and fourth lines result in $n-p-\ell$ equations not involving points in $[0,a]$).

We will assume that $L[X] = 0$ with conditions **a)**, **b)** or **c)** has a unique solution. In this situation, we will say that the couple a, b satisfies Assumption **(A)**.

We shall use in the proof of the next theorem the following known result:

Lemma 3.1. *Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}, \mathcal{F}'_1, \mathcal{F}'_2$ be σ-fields such that $\mathcal{F}'_1 \subset \mathcal{F}_1 \vee \mathcal{G}$ and $\mathcal{F}'_2 \subset \mathcal{F}_2 \vee \mathcal{G}$. Then,*

$$\mathcal{F}_1 \underset{\mathcal{G}}{\perp\!\!\!\perp} \mathcal{F}_2 \quad \Rightarrow \quad \mathcal{F}'_1 \underset{\mathcal{G}}{\perp\!\!\!\perp} \mathcal{F}'_2 \ .$$

We are now ready to prove the following

Theorem 3.2. *Under* **(H)**, *the solution $Y = \{Y(t),\ t \in I\}$ of (2.3)-(2.2) satisfies the conditional independence property* **(P)**, *for $a, b \in I$ verifying Assumption* **(A)**.

Proof of Theorem 3.2: Let us define the σ-fields

$$\mathcal{F}^i_{a,b} = \sigma\{W_t - W_a, t \in [a,b]\}$$

$$\mathcal{F}^e_{a,b} = \sigma\{W_t, t \in [0,a]\} \vee \sigma\{W_1 - W_t, t \in [b,1]\}$$

for $0 \le a < b \le 1$. Notice that $\mathcal{F}^i_{a,b}$ and $\mathcal{F}^e_{a,b}$ are independent.

We shall divide the proof into several steps.

Step 1 *Denote $\mathcal{G}_{a,b} = \sigma\{Y(a), Y(b)\}$. If*

$$\mathcal{F}^i_{a,b} \underset{\mathcal{G}_{a,b}}{\perp\!\!\!\perp} \mathcal{F}^e_{a,b} \quad , \tag{3.7}$$

then (3.3) holds.

Proof of Step 1: It is immediate to prove that $\sigma\{Y(t),\ t \in [a,b]\} \subset \mathcal{G}_{a,b} \vee \mathcal{F}^i_{a,b}$, and $\sigma\{Y(t),\ t \in]a,b[^c\} \subset \mathcal{G}_{a,b} \vee \mathcal{F}^e_{a,b}$. We apply then Lemma 3.1.

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From now on we shall assume that $[a,b] \subset I$ is fixed and such that $[a,b] \cap \{t_1, \dots, t_m\} = \emptyset$, and that Assumption **(A)** is satisfied by a, b. Our goal is to prove (3.7).

Step 2 *Let ℓ and p be as in (3.6). Denote by $\widetilde{Y}$ the solution to (2.3)-(2.2), and define*

$$T^1 := \left(\widetilde{Y}_1(b), \dots, \widetilde{Y}_{n-\ell}(b), \widetilde{Y}_{n-\ell+1}(a), \dots, \widetilde{Y}_n(a)\right) \quad ,$$

$$T^2 := \left(\widetilde{Y}_1(a), \dots, \widetilde{Y}_{n-\ell}(a), \widetilde{Y}_{n-\ell+1}(b), \dots, \widetilde{Y}_n(b)\right) \quad .$$

Then, there exist two functions $g_1, g_2 : I\!\!R^n \times \Omega \to I\!\!R^n$, measurable with respect to $\mathcal{B}(I\!\!R^n) \otimes \mathcal{F}^i_{a,b}$ and $\mathcal{B}(I\!\!R^n) \otimes \mathcal{F}^e_{a,b}$ respectively, and such that

$$T^2 = g_1(T^1, \omega) \qquad \textit{and} \qquad T^1 = g_2(T^2, \omega) \quad . \tag{3.8}$$

Proof of Step 2: Consider equation (2.3) with the boundary conditions

$$\begin{cases} Y_i(b) = T^1_i \quad , \quad i = 1, \dots, n-\ell \\ Y_i(a) = T^1_i \quad , \quad i = n-\ell+1, \dots, n \end{cases} \tag{3.9}$$

on $[a,b]$. The process $\widetilde{Y}$ trivially satisfies (2.3)-(3.9) on $[a,b]$. By the uniqueness Assumption **(A)**, the vector T^2 is determined by T^1 and the increments of the Wiener process in $[a,b]$.

Moreover, the function $g_1(y,\omega)$ so defined has a sense for every $y \in I\!\!R^n$, because we have that the solution to (2.3)-(3.9) is unique, and this fact does not depend on the particular right-hand sides (see **(H)**).

We want to prove analogously the existence of a function g_2 defined by the solution to (2.3) on $]a,b[^c$ with a different set of lateral conditions. Suppose that $0 \le \dots < t_k < a < b < t_{k+1} < \dots \le 1$. Consider first (2.3) on $[0,a]$ with conditions

$$\begin{cases} Y_i(a) = T^2_i \quad , \quad i = 1, \dots, n-\ell \\ \displaystyle\sum_{j=1}^{k} \alpha_{ij}\, Y_n(t_j) = c_i \quad , \quad i = 1, \dots, \ell \quad . \end{cases} \tag{3.10}$$

The restriction to $[0,a]$ of the solution $\widetilde{Y}$ to (2.3)-(2.2) solves (2.3)-(3.10). Consider now the problem (2.3) on $[b,1]$, with conditions

$$\begin{cases} Y_i(b) = T_i^2 \quad , \quad i = n-\ell+1,\dots,n \\ \sum_{j=k+1}^{m} \alpha_{ij}\, Y_n(t_j) = c_i \quad , \quad i = n-p+1,\dots,n \end{cases} \tag{3.11}$$

and the $n-p-\ell$ equations on $[b,1]$ that result from

$$\begin{cases} \sum_{j=1}^{m} \alpha_{ij}\, Y_n(t_j) = c_i \quad , \quad i = \ell+1,\dots,n-p \\ Y_n(t_j) = \tilde{Y}_n(t_j) \quad , \quad j = 1,\dots,k \quad . \end{cases} \tag{3.12}$$

Again, $\widetilde{Y}$ restricted to $[b,1]$ is its unique solution. Therefore, T^1 is determined by T^2 and the increments of the Wiener process W in $]a,b[^c$. As before, the function $g_2(x,\omega)$ so defined has a sense for all $x \in I\!R^n$.

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Step 3 *The functions g_1 and g_2 above satisfy (H.1).*
Proof of Step 3: The solution to a linear differential equation depends linearly on the lateral data c (see (2.10)). Therefore, for each ω fixed, the system (3.8) is linear and it is enough to check that it has a unique solution for $\xi = \eta = 0$.
By arguments similar to those of Step 2, a point

$$(y_1(b),\dots,y_{n-\ell}(b),y_{n-\ell+1}(a),\dots,y_n(a)) \in I\!R^n$$

determines through g_1 a sample path on $[a,b]$ and, in particular, the point

$$(y_1(a),\dots,y_{n-\ell}(a),y_{n-\ell+1}(b),\dots,y_n(b)) \quad .$$

This point in turn determines through g_2 a sample path on $]a,b[^c$ and a point

$$(\overline{y}_1(b),\dots,\overline{y}_{n-\ell}(b),\overline{y}_{n-\ell+1}(a),\dots,\overline{y}_n(a)) \quad ,$$

which coincides with

$$(y_1(b),\dots,y_{n-\ell}(b),y_{n-\ell+1}(a),\dots,y_n(a)) \quad .$$

Hence, the resulting sample path on $[0,1]$ is continuous and satisfies (2.3)-(2.2). Therefore, there can be only one solution (T_1,T_2) to (3.8). The existence is obvious.

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Step 4 *g_1 and g_2 satisfy (H.2)*

Proof of Step 4: As a preliminary result, we will see that for an initial value problem

$$\begin{cases} DY(t) + A(t)\,Y(t) = \dot{B}(t) \quad , \quad t \in I \\ Y(0) = c \quad , \end{cases}$$

the random vector $Y(t)$ $(t > 0)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$. Indeed, in this case

$$Y(t) = \left[\Phi^t(0)\right]^{-1} \left[c + \int_0^t \begin{pmatrix} \Phi_{11}^s(0)\, dW_s \\ \cdots \\ \cdots \\ \Phi_{n1}^s(0)\, dW_s \end{pmatrix} \right]$$

and it is enough to prove the absolute continuity of

$$\left(\int_0^t \Phi_{11}^s(0)\, dW_s, \ldots, \int_0^t \Phi_{n1}^s(0)\, dW_s \right) .$$

Suppose this is not true. Since the vector is centered Gaussian, its law will be supported by a linear subspace of dimension less than n. This means that for some coefficients $b_1, \ldots, b_n$, not all of them equal to zero,

$$\sum_{i=1}^n b_i \int_0^t \Phi_{i1}^s(0)\, dW_s = 0 \quad ,$$

which implies

$$\sum_{i=1}^n b_i\, \Phi_{i1}^s(0) = 0 \quad , \quad \forall\, s \in [0, t] \quad . \tag{3.13}$$

Differentiating (3.13), we get

$$\sum_{i=1}^n b_i\, D\Phi_{i1}^s(0) = 0 \quad , \quad \forall\, s \in [0, t] \quad . \tag{3.14}$$

On the other hand, since $\Phi^s(0) = [\Phi^0(s)]^{-1}$, we have

$$D\Phi^s(0) = -[\Phi^0(s)]^{-1}\, D\Phi^0(s)\, [\Phi^0(s)]^{-1} = \Phi^s(0)\, A(s) \tag{3.15}$$

where A is the matrix defined in (2.4). From (3.15), (3.14) and (3.13), we obtain that

$$\left(0, \sum_{i=1}^n b_i\, \Phi_{i2}^s(0), \ldots, \sum_{i=1}^n b_i\, \Phi_{in}^s(0)\right) \begin{bmatrix} a_{n-1}(s) \\ -1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} = 0 \quad ,$$

from which

$$\sum_{i=1}^{n} b_i \; \Phi_{i2}^{s}(0) \; = \; 0 \quad , \quad \forall \; s \in [0,t] \quad . \tag{3.16}$$

Differentiating now (3.16), and using again (3.15), we also get that

$$\sum_{i=1}^{n} b_i \; \Phi_{i3}^{s}(0) \; = \; 0 \quad , \quad \forall \; s \in [0,t] \quad .$$

Recursively, we arrive to

$$(b_1, \dots, b_n) \; \Phi^s(0) \; = \; (0, \dots, 0) \quad ,$$

which implies that $\Phi^s(0)$ is singular, a contradiction. For an initial value condition $Y(t_0) = c$, a similar proof shows that $Y(t)$ is absolutely continuous for all $t \neq t_0$.

We consider now the boundary value problem that defines g_1: Equation (2.3) on $[a, b]$ with $T^1 = (y_1, \dots, y_n)$ fixed. We want to see that

$$T^2 = (Y_1(a), \dots, Y_{n-\ell}(a), Y_{n-\ell+1}(b), \dots, Y_n(b))$$

is an absolutely continuous random vector. We reason by contradiction. Suppose there is a point $z \in I\!\!R^n$ and a ball $B_\varepsilon(z)$ of radius ε centered at z such that

$$P\big\{\big(Y_1(a), \dots, Y_{n-\ell}(a), Y_{n-\ell+1}(b), \dots, Y_n(b)\big) \in B_\varepsilon(z)\big\} = 0 \quad . \tag{3.17}$$

For any initial condition at a point $\alpha \in]a, b[$, we know that we can find $\omega_0 \in C(I)$ such that

$$Y(a)(\omega_0) = (\bar{z}_1, \dots, \bar{z}_{n-\ell}, \bar{y}_{n-\ell+1}, \dots, \bar{y}_n) \in B_\eta(z_1, \dots, z_{n-\ell}, y_{n-\ell+1}, \dots, y_n)$$

and

$$Y(b)(\omega_0) = (\bar{y}_1, \dots, \bar{y}_{n-\ell}, \bar{z}_{n-\ell+1}, \dots, \bar{z}_n) \in B_\eta(y_1, \dots, y_{n-\ell}, z_{n-\ell+1}, \dots, z_n) \quad ,$$

for any $\eta > 0$. This implies that with the boundary condition

$$(Y_1(b), \dots, Y_{n-\ell}(b), Y_{n-\ell+1}(a), \dots, Y_n(a)) = (\bar{y}_1, \dots, \bar{y}_n) \quad ,$$

and taking $\eta < \frac{\varepsilon}{2}$, we get

$$(Y_1(a), \dots, Y_{n-\ell}(a), Y_{n-\ell+1}(b), \dots, Y_n(b))(\omega_0) = (\bar{z}_1, \dots, \bar{z}_n) \in B_\varepsilon(z) \quad .$$

Taking into account the continuity of the solutions with respect to the boundary data, starting with

$$\big(Y_1(b), \dots, Y_{n-\ell}(b), Y_{n-\ell+1}(a), \dots, Y_n(a)\big) = (y_1, \dots, y_n) \quad ,$$

and for η small enough, we obtain also

$$\big(Y_1(a), \ldots, Y_{n-\ell}(a), Y_{n-\ell+1}(b), \ldots, Y_n(b)\big)(\omega_0) \in B_\varepsilon(z) \quad .$$

This fact contradicts (3.17), due to the continuity of $\omega \mapsto Y(t)(\omega)$.

For the function g_2 a similar proof can be given: We consider equation (2.3) on $]a, b[^c$ with $T^2 = (y_1, \ldots, y_n)$ fixed. Assume

$$P\big\{\big(Y_1(b), \ldots, Y_{n-\ell}(b), Y_{n-\ell+1}(a), \ldots, Y_n(a)\big) \in B_\varepsilon(z)\big\} = 0 \quad . \tag{3.18}$$

For any initial conditions at $\alpha \in]t_k, a[$ and $\beta \in]b, t_{k+1}[$, there exists $\omega_0 \in C(I)$ such that solving (2.3) in $[\alpha, a]$ and $[b, \beta]$, we get

$$Y(a)(\omega_0) = (\bar{y}_1, \ldots, \bar{y}_{n-\ell}, \bar{z}_{n-\ell+1}, \ldots, \bar{z}_n) \in B_\eta(y_1, \ldots, y_{n-\ell}, z_{n-\ell+1}, \ldots, z_n)$$

and

$$Y(b)(\omega_0) = (\bar{z}_1, \ldots, \bar{z}_{n-\ell}, \bar{y}_{n-\ell+1}, \ldots, \bar{y}_n) \in B_\eta(z_1, \ldots, z_{n-\ell}, y_{n-\ell+1}, \ldots, y_n) \quad ,$$

for any $\eta > 0$, and we can finish as for the case of g_1.

Finally, the random vectors $x - g_1(y, \omega)$ and $y - g_2(x, \omega)$ are independent and have the form $x - M_1 y + Z_1(\omega)$ and $y - M_2 x + Z_2(\omega)$ respectively, for some matrices M_1 and M_2 and some Gaussian absolutely continuous vectors Z_1 and Z_2. We deduce that the $\mathbb{R}^{2n}$-valued random vector $(x - g_1(y, \omega), y - g_2(x, \omega))$ has a density which is uniformly bounded in x and y. It follows at once that the function $\delta(x, y)$ in **(H2)** is locally bounded.

◇

Step 5 g_1 *and* g_2 *satisfy (H.3). Precisely,*

$$\det\Big[\mathrm{Id} - \nabla g_1(y, \omega)\ \nabla g_2(x, \omega)\Big]$$

is a constant different from zero.

Proof of Step 5: g_1 and g_2 are affine functions of the first argument, with a non-random linear coefficient (see (2.10)). Therefore, $\nabla g_1(y, \omega)$ and $\nabla g_2(x, \omega)$ are constant matrices, which we denote ∇g_1 and ∇g_2. We have seen that the linear system

$$\begin{cases} x = g_1(y, \omega) \\ y = g_2(x, \omega) \end{cases}$$

admits a unique solution. This is equivalent to

$$\det\Big[\mathrm{Id} - \nabla g_1\ \nabla g_2\Big] = \det\begin{pmatrix} \mathrm{Id} & -\nabla g_1 \\ -\nabla g_2 & \mathrm{Id} \end{pmatrix} \neq 0 \quad .$$

◇

Conclusion We can apply Theorem 3.1 and the factorization in (ii) trivially holds. We deduce the relation (3.7) and, by Step 1, that the process

$$\left\{Y(t) = \left(D^{n-1}X(t), \ldots, X(t)\right) : t \in I\right\}$$

verifies property $(\mathbf{P})$, for $a, b \in I$ satisfying Assumption $(\mathbf{A})$.

◇

Remark 3.1. *We conceive the result of Theorem 3.2 as a first step towards the analysis of the Markov-type properties of linear SDE subject to a general lateral condition $\Lambda[X] = \xi$, with Λ a linear functional on C^{n-1} and a possibly random datum ξ. In this sense it should be pointed out that Theorem 3.2 is still not optimal. For instance, consider the trivial problem*

$$\begin{cases} DX(t) = \dot{W}(t) \\ X(\frac{1}{2}) = 0 \quad . \end{cases} \tag{3.19}$$

The solution $X(t) = -W(\frac{1}{2}) + W(t)$ is in fact a Markov field. Even more, it is a Markov process.

A study of the form that must have the operator Λ to turn the solution of $L[X] = \dot{W}$ into a Markov process has been carried out by Russek [9].

◇

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