

Continuous time random walks under Markovian resettingVicenç Méndez ¹, Axel Masó-Puigdellosas ¹, Trifce Sandev ^{2,3,4} and Daniel Campos¹¹*Grup de Física Estadística, Departament de Física, Facultat de Ciències, Edifici Cc., Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain*²*Research Center for Computer Science and Information Technologies, Macedonian Academy of Sciences and Arts, Bul. Krste Misirkov 2, 1000 Skopje, Macedonia*³*Institute of Physics and Astronomy, University of Potsdam, D-14776 Potsdam-Golm, Germany*⁴*Institute of Physics, Faculty of Natural Sciences and Mathematics, Ss. Cyril and Methodius University, Arhimedova 3, 1000 Skopje, Macedonia*

(Received 2 November 2020; revised 11 January 2021; accepted 15 January 2021; published 3 February 2021)

We investigate the effects of Markovian resetting events on continuous time random walks where the waiting times and the jump lengths are random variables distributed according to power-law probability density functions. We prove the existence of a nonequilibrium stationary state and finite mean first arrival time. However, the existence of an optimum reset rate is conditioned to a specific relationship between the exponents of both power-law tails. We also investigate the search efficiency by finding the optimal random walk which minimizes the mean first arrival time in terms of the reset rate, the distance of the initial position to the target, and the characteristic transport exponents.

DOI: [10.1103/PhysRevE.103.022103](https://doi.org/10.1103/PhysRevE.103.022103)**I. INTRODUCTION**

Diffusion with stochastic resetting was originally proposed some years ago [1]. In this paper a Brownian motion in an infinite medium is interrupted by reset events, which instantaneously returns the particle to the origin. The overall phenomena consist of two random processes independent of each other: the random motion of the particle and the resetting mechanism. Resets happen randomly in time according to a Poisson point process with definite intensity or constant rate. This resetting process is Markovian, and the reset times are exponentially distributed. Actually, the distribution of reset times may be considered, in general, as a distribution with finite moments where the first moment is precisely the inverse of the reset rate. The interest in this kind of problem essentially resides on two rather remarkable facts. First, the verification that resetting stabilizes the random walk process, in the sense that a nonstationary process, as is the diffusion in an infinite medium, becomes stationary when it is affected by the Markovian resetting mechanism. Second, the fact that Markovian resetting may significantly reduce the mean first-passage time which, in turn, may yield a process with infinite first-passage time to reach the target in a finite time. Many generalizations of the random walk, beyond the Brownian dynamics, have been proposed considering Markovian resets. For example, random walks in bounded domains [2–4], subdiffusion, or superdiffusion in continuous space [5–7], superdiffusion in discrete space [8], telegraphic random walks [9], noninstantaneous returns [10–15], residence waiting times after resetting [16], or diffusion in a potential landscape [17]. Other generalizations have been proposed by non-Markovian resetting events which may destroy the two interesting properties mentioned above. Indeed, power-law

distribution of reset times have proved the nonexistence of neither stationary state nor finite mean first-passage time [5]. Other works deal with spatial [18] or temporal [19,20] dependence of the resetting process.

Here we explore another generalization of diffusion with Markovian resetting, wherein the diffusive process between resets is substituted with a process generated through the continuous time random walk (CTRW) scheme. In contrast to recent works that combine Markovian resets with CTRW with finite-moment waiting times and power-law distributions for jump lengths [6], power law distributed waiting times, and jump lengths with finite moments [7,21] or finite-moment waiting times and power-law distributions with drift [22], we assume that both waiting time and jump length probability density functions (PDFs) are distributed according to power laws.

Unlike for superdiffusive transport, it is known that there is no optimum reset rate for the mean first arrival time (MFAT) of subdiffusive transport [5]. Due to the long tails of the waiting time PDF in subdiffusive transport the resetting process always increases the MFAT, and an optimum reset rate is never found. However, the resetting process always helps the long tails in superdiffusive transport to reduce the MFAT and get an optimum reset rate.

Due to the trade-off between both heavy tails, the existence of an optimum reset rate is not obvious which minimizes the MFAT. We find the existence conditions and how the optimum reset rate and the optimum MFAT scale with the initial distance to the target. We find an exact analytic solution to the nonequilibrium stationary state (NESS) and the scaling property that its tail follows. The most efficient random walk (the random walk which minimizes the MFAT) is also studied in terms of the initial distance to the target.

The paper is organized as follows. In Sec. II, we present the CTRW formalism and derive a general expression for the survival probability of arriving at a given position. In Sec. III, we consider resetting on the motion described by the CTRW model and calculate the NESS, the MFAT, and study its optimality. We conclude the paper in Sec. IV.

II. CTRW AND SURVIVAL PROBABILITY

We consider a particle performing a random walk in continuous time. The particle starts the motion from an initial position x_0 jumping instantaneously to a new position where it waits for a time before proceeding with the next jump. Jump lengths and waiting times are independently identically distributed random variables distributed according to the PDFs $\Phi(z)$ and $\varphi(t)$, respectively. The PDF $P(x, t)$ for the particle position at time t is given by the Montroll-Weiss equation [23],

$$P(k, s) = \frac{p_0(k)[1 - \hat{\varphi}(s)]}{s[1 - \hat{\varphi}(s)\Phi(k)]}, \quad (2.1)$$

where

$$P(k, s) = \int_{-\infty}^{\infty} e^{ikx} \int_0^{\infty} e^{-st} P(x, t) dx dt$$

is the Fourier-Laplace transform of $P(x, t)$ and

$$p_0(k) = \int_{-\infty}^{\infty} e^{ikx} P(x, t = 0) dx$$

is the Fourier transform of the initial condition, and $\hat{\varphi}(s)$ and $\Phi(k)$ are the Laplace and Fourier transforms,

$$\hat{\varphi}(s) = \mathcal{L}[\varphi(t)](s) = \int_0^{\infty} e^{-st} \varphi(t) dt,$$

$$\Phi(k) = \mathcal{F}[\Phi(x)](k) = \int_{-\infty}^{\infty} e^{ikx} \Phi(x) dx,$$

of the waiting time and jump length PDFs, respectively. Rearranging Eq. (2.1) in the form

$$s \left[\frac{1}{\hat{\varphi}(s)} - \Phi(k) \right] P(k, s) = p_0(k) \left[\frac{1}{\hat{\varphi}(s)} - 1 \right], \quad (2.2)$$

after straightforward algebraic manipulations, we can rewrite Eq. (2.2) as

$$sP(k, s) - p_0(k) = K(s)[\Phi(k) - 1]P(k, s), \quad (2.3)$$

where we have defined the memory kernel,

$$\hat{K}(s) = \frac{s\hat{\varphi}(s)}{1 - \hat{\varphi}(s)}. \quad (2.4)$$

By inverting Eq. (2.3) in Fourier-Laplace we obtain the so called generalized CTRW master equation [24],

$$\frac{\partial P}{\partial t} = \int_0^t K(t-t') \left[\int_{-\infty}^{\infty} P(x-z, t') \Phi(z) dz - P(x, t') \right] dt'. \quad (2.5)$$

We are interested in power-law PDFs for waiting time $\varphi(t) \sim t^{-(1+\gamma)}$ and jump lengths $\Phi(x) \sim |x|^{-(1+\alpha)}$ which in the Laplace and Fourier spaces read

$$\hat{\varphi}(s) = \frac{1}{1 + (s\tau)^\gamma}, \quad (2.6)$$

with $0 < \gamma < 1$ and

$$\Phi(k) \simeq 1 - \sigma^\alpha |k|^\alpha, \quad (2.7)$$

with $1 < \alpha < 2$, respectively. Then $\hat{K}(s) = s^{1-\gamma}/\tau^\gamma$, and Eq. (2.2) takes the form

$$\tau^\gamma s^\gamma P(k, s) - \tau^\gamma s^{\gamma-1} p_0(k) = -\sigma^\alpha |k|^\alpha P(k, s). \quad (2.8)$$

By inverting in Fourier-Laplace we find the following fractional transport equation (by defining the generalized diffusion coefficient $D = \sigma^\alpha/\tau^\gamma$):

$$\frac{\partial^\gamma P}{\partial t^\gamma} = D \frac{\partial^\alpha P}{\partial |x|^\alpha}, \quad (2.9)$$

where the Laplace transform of the Caputo fractional derivative,

$$\mathcal{L} \left[\frac{\partial^\gamma P(x, t)}{\partial t^\gamma} \right] (s) = s^\gamma P(x, s) - s^{\gamma-1} P(x, t = 0),$$

and the Fourier transform of the Riesz fractional derivative,

$$\mathcal{F} \left[\frac{\partial^\alpha P(x, t)}{\partial |x|^\alpha} \right] (k) = -|k|^\alpha P(k, s)$$

are introduced. The mean squared displacement that characterizes the transport regime scales with time as $\langle x^2(t) \rangle \sim t^{2\gamma/\alpha}$ [25] exhibiting normal diffusion ($2\gamma/\alpha = 1$) and superdiffusion ($2\gamma/\alpha > 1$) of subdiffusion ($2\gamma/\alpha < 1$).

Here we note that the waiting time (2.6) in the time domain has the form [26,27]

$$\varphi(t) = \mathcal{L}^{-1} \left[\frac{1}{1 + (s\tau)^\gamma} \right] = \frac{1}{\tau} \left(\frac{t}{\tau} \right)^{\gamma-1} E_{\gamma, \gamma}(-[t/\tau]^\gamma), \quad (2.10)$$

where $E_{\mu, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + \beta)}$ is the two parameter Mittag-Leffler function [28], which the Laplace transform reads

$$\mathcal{L}[t^{\beta-1} E_{\mu, \beta}(-\omega t^\mu)] = \frac{s^{\mu-\beta}}{s^\mu + \omega}. \quad (2.11)$$

By using the asymptotic expansion of the two parameter Mittag-Leffler function for $z > 1$, see, for example [29],

$$E_{\mu, \beta}(-z) \sim - \sum_{n=1}^{\infty} \frac{(-z)^{-n}}{\Gamma(\beta - \mu n)}, \quad (2.12)$$

we find that $\varphi(t) \sim t^{-\gamma-1}$ for $t/\tau \gg 1$. Moreover, for $t/\tau \ll 1$ the waiting time behaves as $\varphi(t) \sim t^{\gamma-1}$. The non-negativity of the waiting time can be shown from Eq. (2.6) by using the Bernstein theorem [30]. Moreover, one can show that the waiting time (2.10) is completely a monotone function since $e_{\mu, \beta}(t) = t^{\beta-1} E_{\mu, \beta}(-t^\mu)$ is completely monotone for $0 < \mu \leq \beta \leq 1$ with $0 < \mu \leq 1$, see Refs. [29,31].

Regarding the jump length PDF (2.7), which is a jump length PDF for Lévy flights, we consider only the case with $1 < \alpha \leq 2$ since the first arrival density for a Lévy search for a point target, for example, the origin, vanishes for $0 < \alpha \leq 1$ [32].

To compute the MFAT under Markovian resetting it is necessary first to find the survival probability $Q_{x_0}(t)$ up to time t of the transport process. In particular, $Q_{x_0}(t)$ is the probability of not having reached the origin ($x = 0$) in the first trip, which ends at a random time t' , when the particle starts the motion

at position x_0 . Then the first arrival time probability density $f_a(x_0, t)$ at the origin at a time t for a particle that starts its motion at position x_0 and the survival probability are related to each other via

$$Q_{x_0}(t) = 1 - \int_0^t f_a(x_0, t') dt', \quad (2.13)$$

or equivalently by $f_a(x_0, t) = -\partial Q_{x_0}(t)/\partial t$. We follow the method in Ref. [32] since it has the advantage of being simpler than the traditional procedure of solving the equation for the propagator with an absorbing boundary at x_0 . Furthermore, it has been shown to be valid even for Lévy flights [32,33]. We make use of the generalized master equation, which is written as a rate equation for the probability with a δ sink of a strength $f_a(x_0, t)$,

$$\frac{\partial P(x, t)}{\partial t} = \int_0^t K(t-t') \left[\int_{-\infty}^{\infty} P(x-z, t') \Phi(z) dz - P(x, t') \right] dt' - f_a(x_0, t) \delta(x), \quad (2.14)$$

where $P(x, t)$ is now a non-normalized probability. We assume that at $t = 0$ the particle is placed at $x = x_0$, i.e., $P(x, t = 0) = p_0(x) = \delta(x - x_0)$. Taking the Fourier-Laplace transform of Eq. (2.14) and $p_0(k) = e^{ikx_0}$ one gets

$$P(k, s) = \frac{e^{ikx_0}}{s + \hat{K}(s)[1 - \Phi(k)] - f_a(x_0, s) \frac{1}{s + \hat{K}(s)[1 - \Phi(k)]}}. \quad (2.15)$$

Since $f_a(x_0, t)$ measures the first arrival time at the origin, we have taken into account the δ sink in Eq. (2.14), i.e., the origin is a perfectly absorbing boundary and $P(x = 0, t) = 0$. We can exploit this property by arguing that the inverse Fourier

transform of $P(k, s)$ is

$$P(x, s) = \mathcal{F}^{-1}[P(k, s)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} P(k, s) dk, \quad (2.16)$$

and taking $x = 0$ we have

$$\int_{-\infty}^{\infty} P(k, s) dk = 0. \quad (2.17)$$

Integrating Eq. (2.15) and taking into account Eq. (2.17) we obtain

$$\begin{aligned} \hat{Q}_{x_0}(s) &= \frac{1}{s} [1 - f_a(s)] \\ &= \frac{1}{s} \left[1 - \frac{P(x=0, s; x_0)}{P(x=0, s; 0)} \right], \end{aligned} \quad (2.18)$$

once we have transformed Eq. (2.13) by Laplace. Note that the propagator $P(x, s; x_0)$ is given from Eq. (2.8) by

$$P(x, s; x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ik(x-x_0)}}{s + \hat{K}(s)[1 - \Phi(k)]} dk. \quad (2.19)$$

Considering (2.6) and (2.7) in (2.19), the propagator reads

$$P(x, s; x_0) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos[k(x-x_0)] dk}{s + s^{1-\gamma} \tau^{-\gamma} (\sigma k)^\alpha},$$

which inserted in (2.18) allows to find

$$\hat{Q}_{x_0}(s) = \frac{\alpha \sigma \sin\left(\frac{\pi}{\alpha}\right)}{\pi (s\tau)^{\gamma/\alpha}} \int_0^{\infty} \frac{1 - \cos(kx_0)}{s + s^{1-\gamma} \tau^{-\gamma} (\sigma k)^\alpha} dk, \quad (2.20)$$

where we made use of the result,

$$\int_0^{\infty} \frac{dk}{(s\tau)^{\gamma/\alpha} \sigma^\alpha + k^\alpha} = \frac{\pi \sigma^{\alpha-1}}{\alpha \sin(\pi/\alpha)} (s\tau)^{\gamma[(1/\alpha)-1]}. \quad (2.21)$$

Inverting Eq. (2.20) by Laplace we find

$$\begin{aligned} Q_{x_0}(t) &= 1 - \frac{\alpha \sigma \sin\left(\frac{\pi}{\alpha}\right)}{\pi} (t/\tau)^{\gamma/\alpha} \int_0^{\infty} \cos(kx_0) E_{\gamma, 1+(\gamma/\alpha)} \left(-\sigma^\alpha k^\alpha \frac{t^\gamma}{\tau^\gamma} \right) dk \\ &= 1 - \sin\left(\frac{\pi}{\alpha}\right) H_{3,3}^{2,1} \left[\frac{|x_0|/\sigma}{(t/\tau)^{\gamma/\alpha}} \left| \begin{matrix} (\frac{\alpha-1}{\alpha}, \frac{1}{\alpha}) & (1, \frac{\gamma}{\alpha}) & (\frac{1}{2}, \frac{1}{2}) \\ (0, 1) & (\frac{\alpha-1}{\alpha}, \frac{1}{\alpha}) & (\frac{1}{2}, \frac{1}{2}) \end{matrix} \right. \right], \end{aligned} \quad (2.22)$$

where $H_{p,q}^{m,n}(z)$ is the Fox-H function [34]. Known particular cases may be recovered from (2.22). For $\gamma = 1$ and $\alpha = 2$, the transport process corresponds to normal diffusion. Inserting these values in (2.22) and taking into account that in this case,

$$E_{1,3/2}(z) = \frac{e^z}{\sqrt{z}} \operatorname{erf}(\sqrt{z}),$$

the integral in (2.22) can be straightforwardly computed to get

$$Q_{x_0}(t) = \operatorname{erf}\left(\frac{|x_0|}{2\sigma} \sqrt{\frac{\tau}{t}}\right) = \operatorname{erf}\left(\frac{|x_0|}{\sqrt{4Dt}}\right), \quad (2.23)$$

which corresponds to the known result in Ref. [35] by considering $D = \sigma^2/\tau$. If we set $\gamma = 1$ in (2.22) we recover the result found in Ref. [32] for Lévy flights. The large time behavior of $Q_{x_0}(t)$ can be found by using the series expansion of the Fox function in Eq. (2.19) for $t \rightarrow \infty$ (small argument

expansion of the Fox function), to get [34]

$$\begin{aligned} Q_{x_0}(t) &\approx \frac{\alpha \sqrt{\pi}}{2^\alpha \Gamma(\frac{\alpha}{2}) \Gamma(\frac{1+\alpha}{2})} \left(\frac{|x_0|}{\sigma}\right)^{\alpha-1} \frac{(t/\tau)^{-(\gamma/\alpha)(\alpha-1)}}{\Gamma(1-\gamma+\frac{\gamma}{\alpha})} \\ &\sim \frac{1}{t^{(\gamma/\alpha)(\alpha-1)}}. \end{aligned} \quad (2.24)$$

Introducing $\alpha = 2$ in (2.24) with $0 < \gamma < 1$ (subdiffusion) and $\gamma = 1$ with $1 < \alpha < 2$ (Lévy flights) we recover the known results $Q_{x_0}(t) \sim t^{-\gamma/2}$ [36] and $Q_{x_0}(t) \sim t^{-1+1/\alpha}$ [32], respectively. From Eq. (2.13) the long term behavior of the first arrival time PDFs in these two cases are $f_a(x_0, t) \sim t^{-\gamma/2-1}$ and $f_a(x_0, t) \sim t^{-2+1/\alpha}$, respectively. For computing purposes it is interesting to express the Fox function in Eq. (2.22) as a power series. By using Ref. [34] we find after

some simplifications,

$$\begin{aligned} Q_{x_0}(t) = \sin\left(\frac{\pi}{\alpha}\right) & \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin\left(\frac{\pi(1+n)}{2}\right) \left(\frac{|x_0|}{\sigma}\right)^n}{n! \sin\left(\frac{\pi(1+n)}{\alpha}\right) \Gamma\left(1 - \frac{\gamma n}{\alpha}\right) \left(\frac{t}{\tau}\right)^{\gamma n/\alpha}} \right. \\ & \left. + \frac{\alpha}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{2^{-\alpha n} (-1)^n \Gamma\left(\frac{1-n\alpha}{2}\right) \left(\frac{|x_0|}{\sigma}\right)^{n\alpha-1}}{\Gamma\left(\frac{n\alpha}{2}\right) \Gamma\left(1 + \frac{\gamma}{\alpha} - \gamma n\right) \left(\frac{t}{\tau}\right)^{(\gamma/\alpha)(n\alpha-1)}} \right]. \end{aligned} \quad (2.25)$$

This is where the general expressions (2.22) and (2.25) and the scaling (2.24) have been reported in such a general form.

III. CTRW AND MARKOVIAN RESETTING

When resetting is taken into account the particle moves according to the CTRW propagator (2.1) during a period called the reset time and then the particle jumps instantaneously to the initial position $x = x_0$ to start its motion again. Then the incorporation of the resetting process to the transport process results in a sequence of transport periods and instantaneous resets to x_0 . The time spent between two consecutive resets is the reset time and is a random variable distributed according to the PDF $\varphi_R(t)$. A general formulation for the combination of a general transport with resetting has been recently studied [5] where the NESS and the MFAT have been obtained for any transport propagator and for any resetting PDF.

The propagator $\rho(x, t; x_0)$ of the joint process (transport and resetting to the initial position x_0) is given by the renewal equation [5],

$$\rho(x, t; x_0) = \varphi_R^*(t) P(x, t; x_0) + \int_0^t \varphi_R(t') \rho(x, t - t'; x_0) dt', \quad (3.1)$$

where $\varphi_R^*(t) = \int_t^\infty \varphi_R(t') dt'$ is the probability of the first reset happening after time t . In the Laplace space this equation has the form

$$\rho(x, s; x_0) = \frac{\mathcal{L}[\varphi_R^*(t) P(x, t; x_0)](s)}{1 - \hat{\varphi}_R(s)}. \quad (3.2)$$

The survival probability $S_{x_0}(t)$ of the joint process (transport and resetting) is given by the renewal equation,

$$S_{x_0}(t) = \varphi_R^*(t) Q_{x_0}(t) + \int_0^t \varphi_R(t') Q_{x_0}(t') S_{x_0}(t - t') dt', \quad (3.3)$$

where $Q_{x_0}(t)$ is the survival probability of the transport process and is given by Eq. (2.18). Equation (3.3) can be solved in the Laplace space,

$$\hat{S}_{x_0}(s) = \frac{\mathcal{L}[\varphi_R^*(t) Q_{x_0}(t)](s)}{1 - \mathcal{L}[\varphi_R(t) Q_{x_0}(t)](s)}. \quad (3.4)$$

The total (transport and resetting) first arrival time probability density at origin $F_a(x_0, t)$ is obtained in terms of $S_{x_0}(t)$: $F_a(x_0, t) = -\partial S_{x_0}(t)/\partial t$. Then the MFAT is given by

$$\begin{aligned} T(x_0) &= \int_0^\infty t F_a(x_0, t) dt = \lim_{s \rightarrow 0} \hat{S}_{x_0}(s) \\ &= \frac{\int_0^\infty \varphi_R^*(t) Q_{x_0}(t) dt}{1 - \int_0^\infty \varphi_R(t) Q_{x_0}(t) dt}. \end{aligned} \quad (3.5)$$

Equation (3.5) then provides the mean time that a particle needs to arrive for the first time at the origin $x = 0$ where a target is located. Likewise, $|x_0|$ is the initial distance between the particle and the target. The particle, which starts its motion at $x = x_0$ [whose survival probability is $Q_{x_0}(t)$] is reset to $x = x_0$ after a random time distributed according to the PDF $\varphi_R(t)$. Then, the MFAT in Eq. (3.5) is a measure of the search efficiency when the particle movement is described by a CTRW under a resetting mechanism.

To obtain specific results for the NESS and the MFAT we need to consider particular expressions for $\varphi_R(t)$. Throughout this paper we consider that resetting is a Markovian process, i.e., the reset rate is exponentially distributed

$$\varphi_R(t) = r e^{-rt}. \quad (3.6)$$

Below we apply these results to the CTRW propagator in Eq. (2.1). Considering Eqs. (3.5) and (3.6) it is found

$$T(x_0) = \frac{Q_{x_0}(s=r)}{1 - r Q_{x_0}(s=r)}, \quad (3.7)$$

where $Q_{x_0}(s=r)$ is found from Eq. (2.20).

A. NESS

Applying the Fourier transform to Eq. (3.2) and introducing the propagator in (2.19) and (3.6), one can obtain the propagator of the joint process in the Fourier-Laplace space to be

$$\rho(k, s; x_0) = \frac{1}{2\pi s} \int_{-\infty}^{\infty} \frac{(r+s)e^{-ik(x-x_0)} dk}{s+r+K(s+r)[1-\Phi(k)]}. \quad (3.8)$$

To obtain the stationary solution of the joint process we take the limit $s \rightarrow 0$ to Eq. (3.8) and inserting Eqs. (2.6) and (2.7) we finally get the exact expression for the NESS,

$$\begin{aligned} \rho(x, x_0) &= \frac{1}{\pi} \int_0^\infty \frac{\cos[k(x-x_0)]}{1+(r\tau)^{-\gamma} \sigma^\alpha k^\alpha} dk \\ &= \frac{1}{\pi} \int_0^\infty \cos[k(x-x_0)] H_{1,1}^{1,1} \left[\frac{\sigma^\alpha k^\alpha}{(r\tau)^\gamma} \middle| \begin{matrix} (0, 1) \\ (0, 1) \end{matrix} \right] dk \\ &= \frac{1}{|x-x_0|} H_{2,3}^{2,1} \left[\frac{(r\tau)^\gamma |x-x_0|^\alpha}{\sigma^\alpha} \middle| \begin{matrix} (1, 1) & (1, \frac{\alpha}{2}) \\ (1, \alpha) & (1, 1) & (1, \frac{\alpha}{2}) \end{matrix} \right]. \end{aligned} \quad (3.9)$$

In Fig. 1 we show a comparison of the NESS obtained in (3.9) with numerical simulations sampling dispersal distances from a Lévy PDF [$\Phi(k) = e^{-\sigma^\alpha |k|^\alpha}$]. It is noticeable that to get analytic results we have considered actually an approximation to the Lévy PDF for large dispersal distances in Eq. (2.7) [$\Phi(k) = e^{-\sigma^\alpha |k|^\alpha} \approx 1 - \sigma^\alpha |k|^\alpha$]. For this reason the agreement between numerical and theoretical results fails close to $x = 0$ (not shown). In panel (a) we show the case of $\gamma = 0.25$ (in red) and $\gamma = 0.75$ (in blue) for fixed $\alpha = 1.25$. We see how the exponent γ modifies the shape of the NESS. In panel (b) we consider $\gamma = 0.25$ and the cases of $\alpha = 1.25$ (in red) and $\alpha = 1.75$ (in blue). The tail decays faster with x as α increases as we also show, in turn, analytically. The Fox function admits series expansion for $|x-x_0| \gg \sigma$ [34].

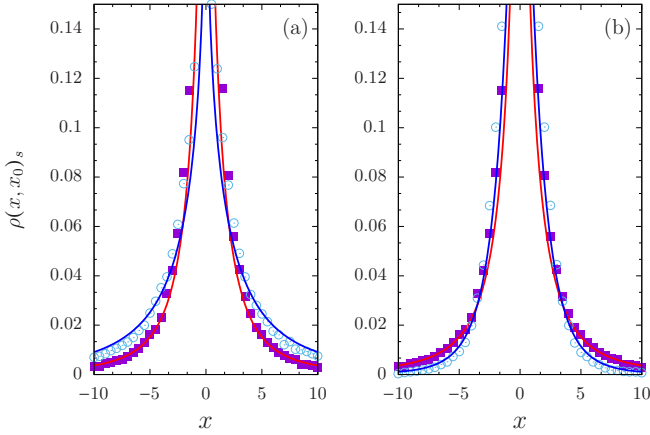


FIG. 1. NESS for different values of α and γ with $\sigma = r = 1$, $\tau = 0.1$, $x_0 = 0$. In panel (a) we fix $\alpha = 1.25$. The cases of $\gamma = 0.25$ and $\gamma = 0.75$ are drawn in red and blue, respectively. In panel (b) we fix $\gamma = 0.25$. The cases of $\alpha = 1.25$ and $\alpha = 1.75$ are drawn in red and blue, respectively. The analytical solution is computed from Eq. (3.9) and shown with solid curves whereas symbols correspond to numerical simulations.

Taking the lowest order we obtain the following scaling for the tail of the NESS:

$$\rho_s(x, x_0) \sim \frac{\sigma^{\alpha+1}}{|x - x_0|^{\alpha+1}}. \quad (3.10)$$

It is interesting to note that this scaling behavior is not affected by the waiting time PDF tail, i.e., the tail of the NESS is controlled by the jump length PDF only.

B. MFAT

The MFAT is found by introducing Eq. (2.20) into Eq. (3.7) to get after some manipulations,

$$T(x_0) = \frac{1}{r} \left[\frac{\pi (r\tau)^{\gamma/\alpha}}{\alpha\sigma \sin(\frac{\pi}{\alpha}) I(x_0, r)} - 1 \right], \quad (3.11)$$

where

$$I(x_0, r) = \int_0^\infty \frac{\cos(kx_0)}{1 + (r\tau)^{-\gamma} \sigma^\alpha k^\alpha} dk \quad (3.12)$$

is defined.

This expression may be evaluated analytically by using Ref. [34],

$$\begin{aligned} I(x_0, r) &= \int_0^\infty \cos(kx_0) H_{1,1}^{1,1} \left[\frac{\sigma^\alpha k^\alpha}{(r\tau)^\gamma} \middle| \begin{matrix} (0, 1) \\ (0, 1) \end{matrix} \right] dk \\ &= \frac{\pi}{\alpha |x_0|} H_{2,3}^{2,1} \left[\frac{(r\tau)^{\gamma/\alpha} |x_0|}{\sigma} \middle| \begin{matrix} (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \\ (1, 1) & (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \end{matrix} \right]. \end{aligned} \quad (3.13)$$

From (3.11) and (3.13) the MFAT takes finally the form

$$T(x_0) = \frac{1}{r} \left\{ \frac{\frac{|x_0|}{\sigma} (r\tau)^{\gamma/\alpha} \left[\sin\left(\frac{\pi}{\alpha}\right) \right]^{-1}}{H_{2,3}^{2,1} \left[\frac{(r\tau)^{\gamma/\alpha} |x_0|}{\sigma} \middle| \begin{matrix} (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \\ (1, 1) & (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \end{matrix} \right]} - 1 \right\}. \quad (3.14)$$

Since there is no additional convergence restriction in calculating the integral in Eq. (3.13) we can assert that the MFAT is finite for any set of parameters γ and α , i.e., for any specific transport process. Equation (3.14) holds for any Markovian resetting (i.e., for a resetting time PDF with finite moments since r is nothing but the inverse of the first moment) with a general CTRW since the expressions for the PDFs in Eqs. (2.6) and (2.7) may lead to subdiffusive, superdiffusive, or normal transports [25]. Making use of the power expansions of the Fox functions [34] we can approximate (3.11) in the limits $r\tau \ll 1$ (small reset rate) and $r\tau \gg 1$ (large reset rate). For $r\tau \ll 1$ the leading terms in Eq. (3.13) are

$$\begin{aligned} I(x_0, r) &\simeq \Gamma(1 - 1/\alpha) \Gamma(1/\alpha) \frac{(r\tau)^{\gamma/\alpha}}{\alpha\sigma} \\ &\quad + \frac{\pi \Gamma(1 - \alpha)}{\Gamma(\alpha/2) \Gamma(1 - \alpha/2)} \frac{(r\tau)^\gamma |x_0|^{\alpha-1}}{\sigma^\alpha}, \end{aligned} \quad (3.15)$$

which can be inserted in (3.11) to get

$$T(x_0) \approx \frac{\tau \alpha \Gamma(2 - \alpha) \sin(\frac{\pi}{\alpha}) \sin(\frac{\pi\alpha}{2})}{\pi(\alpha - 1)(r\tau)^{\gamma[(1/\alpha)-1]+1}} \left(\frac{|x_0|}{\sigma} \right)^{\alpha-1}. \quad (3.16)$$

Analogously, for $r\tau \gg 1$ the leading terms in Eq. (3.13) are

$$I(x_0, r) \simeq \frac{\Gamma(1 + \alpha) \sin(\pi\alpha/2) \sigma^\alpha}{(r\tau)^\gamma |x_0|^{\alpha+1}}, \quad (3.17)$$

which leads us to

$$T(x_0) \approx \frac{\tau \pi (r\tau)^{\gamma[1+(1/\alpha)]-1}}{\alpha \sin(\frac{\pi}{\alpha}) \sin(\frac{\pi\alpha}{2}) \Gamma(1 + \alpha)} \left(\frac{|x_0|}{\sigma} \right)^{1+\alpha}. \quad (3.18)$$

C. Optimal reset rate

The question is now to find the condition whether there is an optimal MFAT and more specifically, what is the set of values of parameters (γ, α) for which there is a reset rate that optimizes the MFAT. To this end we analyze the behavior of $T(x_0)$ in the limits $r \rightarrow 0^+$ and $r \rightarrow \infty$. For fixed x_0 , the MFAT in the limit $r \rightarrow 0^+$ is given in Eq. (3.16), that is, $T(x_0) \sim r^{\gamma[1-(1/\alpha)]-1}$. Since $\gamma \in (0, 1)$ and $\alpha \in (1, 2)$ the exponent is such that $\gamma(1 - 1/\alpha) < 1$, i.e., it is always negative. In consequence, $T(x_0) \rightarrow \infty$ as $r \rightarrow 0^+$. In addition, $T(x_0)$ is a decreasing function with r when r takes small values. On the other hand, when r takes large values the MFAT is given in Eq. (3.18), i.e., $T(x_0) \sim r^{\gamma/\alpha+\gamma-1}$, which is an increasing function of r only if $\gamma > \gamma_c$ with

$$\gamma_c = \frac{\alpha}{\alpha + 1}. \quad (3.19)$$

Moreover, $T(x_0)$ is a monotonically decreasing function of r when r is large if $\gamma < \gamma_c$, which means that even if there is a local minimum for a given r since it is still decreasing with r , then the minimum MFAT is $T(x_0) = 0$. In consequence, there exists a nonzero minimum MFAT if $\gamma \geq \gamma_c$. Note that when $\gamma = \gamma_c$ two different situations could happen: The minimum is either attained at $r \rightarrow \infty$ or for a specific value of r . Since $T(x_0)$ is a decreasing function of r for small r and it tends to a constant value as $r \rightarrow \infty$, if $\partial T(x_0)/\partial r < 0$ the minimum MFAT is attained asymptotically at $r \rightarrow \infty$. Contrarily, if $\partial T(x_0)/\partial r > 0$ for large r the minimum MFAT is attained at

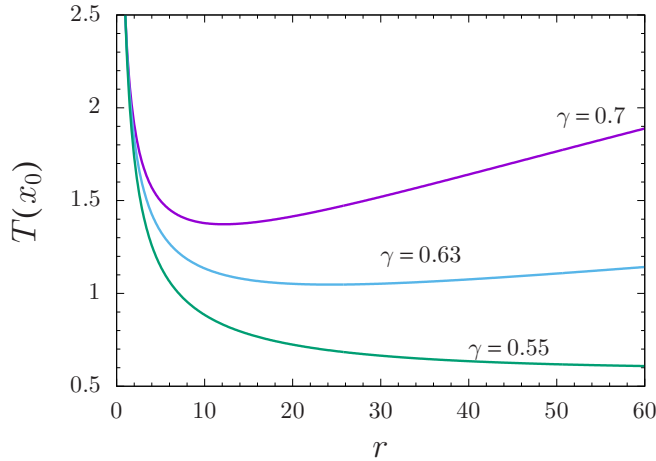


FIG. 2. MFAT computed from Eq. (3.14) for different values of γ . Parameters values are $x_0 = \tau = \sigma = 1$ and $\alpha = 1.75$. $\gamma_c = 0.636$.

a given r . To uncover which is the actual situation we take $\gamma = \gamma_c$ in (3.18) to get

$$T(x_0) \approx \frac{\pi \tau (|x_0|/\sigma)^{1+\alpha}}{\alpha \sin(\pi/\alpha) \Gamma(1+\alpha) \sin(\pi\alpha/2)} - \frac{1}{r}.$$

It is easy to check that $\partial T(x_0)/\partial r \approx 1/r^2 > 0$ so that a local minimum is attained when $\gamma = \gamma_c$. Therefore, we can conclude that there exists an optimum reset rate r which minimizes $T(x_0)$ if and only if,

$$\gamma \geq \gamma_c.$$

This is one of the main results of this paper. In Fig. 2 we plot the MFAT given by (3.14) for different values of γ above, below, and at $\gamma = \gamma_c$. As can be seen for $\gamma = \gamma_c$ there is a local minimum for the MFAT. The optimal reset rate is obtained by solving numerically the equation $\partial T(x_0)/\partial r = 0$. Taking the derivative of (3.14) and using the properties of the derivatives of the Fox functions [34] one finds, after some simplifications, that the optimal reset rate r^* is given by

$$r^* = \frac{1}{\tau} \left[\frac{\sigma}{|x_0|} z(\alpha, \gamma) \right]^{\alpha/\gamma}, \quad (3.20)$$

where $z = z(\alpha, \gamma)$ is the solution to the equation,

$$\begin{aligned} 1 - \frac{\gamma}{\alpha} + \frac{H_{2,3}^{2,1} \left[z \begin{vmatrix} (0, \frac{1}{\alpha}) & (1, \frac{1}{2}) \\ (1, 1) & (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \end{vmatrix} \right]}{H_{2,3}^{2,1} \left[z \begin{vmatrix} (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \\ (1, 1) & (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \end{vmatrix} \right]} \\ = \frac{\sin(\pi/\alpha)}{z} H_{2,3}^{2,1} \left[z \begin{vmatrix} (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \\ (1, 1) & (1, \frac{1}{\alpha}) & (1, \frac{1}{2}) \end{vmatrix} \right]. \end{aligned} \quad (3.21)$$

From (3.20) it is found the scaling dependence $r^* \sim |x_0|^{-\alpha/\gamma}$ which generalizes recent results obtained for any α and $\gamma = 1$ [6]. If (3.20) is introduced in Eq. (3.14) then the

optimum MFAT obeys the scaling relation,

$$T^*(x_0) = \left(\frac{|x_0|}{\sigma} \right)^{\alpha/\gamma} T^*(\sigma), \quad (3.22)$$

which is again a generalization of the scaling found in Ref. [6]. Here, $T^*(x_0)$ grows with x_0 faster than for $\gamma = 1$ due to the effect of the heavy tailed waiting time PDF which slows down the search process of reaching the target at the origin.

D. Optimal random walk

Analogously, fixing the value of the reset rate r , the optimal random walk [characterized by the values of the exponents (γ, α)] which minimizes the MFAT, depends on the distance between the initial position of the random walker x_0 and the target point $x = 0$. Let us consider two limiting situations: x_0 small and large. We assume $x_0 > 0$ for simplicity, otherwise, we should replace x_0 by $|x_0|$ from now on. As can be seen from (3.14) the limit for small x_0 is equivalent to consider $r\tau \ll 1$. So that, when x_0 is small the MFAT is given by (3.16).

As can be checked numerically, the minimum value of $T(x_0)$ is attained when $\alpha = 2$ for the other parameters fixed, so the Brownian motion is the most effective random walk when the particle starts the motion close to the resetting point. The optimal value is

$$T^*(x_0) \approx \tau (|x_0|/\sigma) (r\tau)^{-1+\gamma/2},$$

and is decreasing with r . This means that when the initial position of the particle is close to the target, the search process is more efficient when the particle motion has a short-tailed jump length distribution, being Brownian when $\gamma = 1$.

Analogously, the limit for large x_0 is equivalent to consider $r\tau \gg 1$, and the MFAT is given by (3.18). For fixed r , x_0 , and γ , this expression has a minimum for a given value of the exponent α within the interval $(1, 2)$ which means that Lévy-like jumps are optimal when the initial position of the particle x_0 is far from the target position. Although the prefactors depend explicitly on γ , the scalings in (3.16) and (3.18) of $T(x_0)$ on x_0 only depend on the exponent α as in Ref. [7].

However, as can be shown numerically, the MFAT given in (3.18) has a minimum for a given value of α which depends on x_0 , r , and γ . Let $\alpha_{\text{opt}}(r, x_0, \gamma)$ denote the value of α at which the MFAT is minimum. If the minimum MFAT is attained at $\alpha_{\text{opt}} \in [1, 2)$ then long-tailed jump distributions are optimal for the search strategy, but if it is attained at $\alpha_{\text{opt}} = 2$ then short-tailed jump distributions are the optimal. In order to gain a deeper understanding we can draw a phase diagram for the optimality regions by inspecting the minimum MFAT numerically. More specifically, we have computed Eq. (3.5) for $\sigma = \tau = 1$ and $\gamma = 0.5$ and different values of x_0 and r as shown in Fig. 3. For each set of parameters, we have determined whether the MFAT has a minimum between at $\alpha_{\text{opt}} \in [1, 2)$ or at $\alpha_{\text{opt}} = 2$ and have depicted the frontier.

For parameter values below the critical curve, the MFAT has a minimum at $\alpha_{\text{opt}} = 2$, i.e., short-tailed jump distributions are optimal. For parameter values above the critical curve the minimum MFAT is attained for a value of α between 1 and 2 and, therefore, long-tailed jump distributions attain the optimal MFAT. This, in the particular $\gamma = 1$ case, shows that, on one hand, Brownian motion would be the optimal strategy

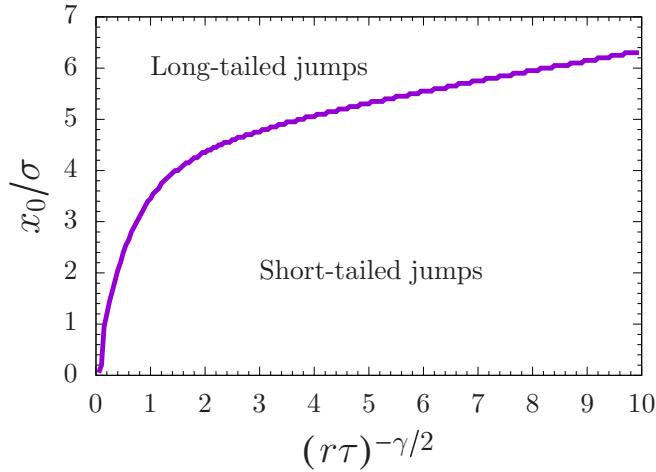


FIG. 3. Optimality regions in the parameter space. We take $\sigma = \tau = 1$ and $\gamma = 0.5$ and vary x_0 and r .

to find a target near the origin. On the other hand, when the target is far from the origin, Lévy flights become the optimal strategy to find it.

IV. CONCLUSIONS

In this paper we considered a random walk with waiting times and jump lengths distributed according to power laws and interrupted by a Markovian resetting process. The walker starts the motion at point x_0 , and a target is assumed to be

located at $x = 0$. First we found the exact solution and the scaling in the long time limit for the survival probability of not having reached the target in the first trip in absence of resetting. Second, we obtained exact analytic solutions to the NESS and the MFAT in the presence of Markovian resetting. Due to the opposite effect of the heavy tails of the waiting times and jump length PDFs, there is a critical value of the waiting time exponent γ for the existence of an optimum MFAT. We have obtained the critical exponent γ and have found the parameter regions for optimal MFAT. We have also determined which type of motion strategy (diffusion or Lévy flight) is optimal depending on the distance x_0 between the target and the resetting positions. Despite the widespread belief that Lévy flight is the optimal strategy for large x_0 we have found that the optimality between Brownian or Lévy strategies not only depends on x_0 , but also depends on the reset rate r . For example, if r is large the Lévy strategy could be the optimal strategy even for small x_0 . We have also found that for any reset rate r , one always finds a transition between optimal Brownian motion and optimal Lévy flight at a critical value of x_0 . This transition is also shown to exist for any choice of the waiting time distribution of the form in Eq. (2.6), including long-tailed waiting times. Therefore, depending on the environmental (x_0, r) and the internal (γ, τ, σ) parameters of the walker, any type of motion from subdiffusion to superdiffusion could be optimal to find the target.

ACKNOWLEDGMENTS

This research was partially supported by Grant No. CGL2016-78156-C2-2-R (V.M., D.C., and A.M.-P.). T.S. was supported by the Alexander von Humboldt Foundation.

-
- [1] M. R. Evans and S. N. Majumdar, Diffusion with Stochastic Resetting, *Phys. Rev. Lett.* **106**, 160601 (2011).
 - [2] C. Christou and A. Schadschneider, Diffusion with resetting in bounded domains, *J. Phys. A: Math. Theor.* **48**, 285003 (2015).
 - [3] A. Pal and V. V. Prasad, First passage under stochastic resetting in an interval, *Phys. Rev. E* **99**, 032123 (2019).
 - [4] X. Durang, S. Lee, L. Lizana, and J.-H. Jeon, First-passage statistics under stochastic resetting in bounded domains, *J. Phys. A: Math. Theor.* **52**, 224001 (2019).
 - [5] A. Masó-Puigdellosas, D. Campos, and V. Méndez, Transport properties and first-arrival statistics of random motion with stochastic reset times, *Phys. Rev. E* **99**, 012141 (2019).
 - [6] Ł. Kuśmierz and E. Gudowska-Nowak, Optimal first-arrival times in Lévy flights with resetting, *Phys. Rev. E* **92**, 052127 (2015).
 - [7] Ł. Kuśmierz and E. Gudowska-Nowak, Subdiffusive continuous-time random walks with stochastic resetting, *Phys. Rev. E* **99**, 052116 (2019).
 - [8] L. Kuśmierz, S. N. Majumdar, S. Sabhapandit, and G. Schehr, First Order Transition for the Optimal Search Time of Lévy Flights with Resetting, *Phys. Rev. Lett.* **113**, 220602 (2014).
 - [9] J. Masoliver, Telegraphic processes with stochastic resetting, *Phys. Rev. E* **99**, 012121 (2019).
 - [10] A. Masó-Puigdellosas, D. Campos, and V. Méndez, Transport properties of random walks under stochastic noninstantaneous resetting, *Phys. Rev. E* **100**, 042104 (2019).
 - [11] A. Pal, Ł. Kuśmierz, and S. Reuveni, Invariants of motion with stochastic resetting and space-time coupled returns, *New J. Phys.* **21**, 113024 (2019).
 - [12] A. Pal, Ł. Kuśmierz, and S. Reuveni, Time-dependent density of diffusion with stochastic resetting is invariant to return speed, *Phys. Rev. E* **100**, 040101(R) (2019).
 - [13] A. Pal, Ł. Kuśmierz, and S. Reuveni, Search with home returns provides advantage under high uncertainty, *Phys. Rev. Res.* **2**, 043174 (2020).
 - [14] A. S. Bodrova and I. M. Sokolov, Resetting processes with noninstantaneous return, *Phys. Rev. E* **101**, 052130 (2020).
 - [15] A. S. Bodrova and I. M. Sokolov, Brownian motion under noninstantaneous resetting in higher dimensions, *Phys. Rev. E* **102**, 032129 (2020).
 - [16] A. Masó-Puigdellosas, D. Campos, and V. Méndez, Stochastic movement subject to a reset-and-residence mechanism: Transport properties and first arrival statistics, *J. Stat. Mech.: Theory Exp.* (2019) 033201.
 - [17] A. Pal, Diffusion in a potential landscape with stochastic resetting, *Phys. Rev. E* **91**, 012113 (2015).

- [18] M. R. Evans and S. N. Majumdar, Diffusion with optimal resetting, *J. Phys. A: Math. Theor.* **44**, 435001 (2011).
- [19] A. Pal, A. Kundu, and M. R. Evans, Diffusion under time-dependent resetting, *J. Phys. A: Math. Theor.* **49**, 225001 (2016).
- [20] V. P. Shkilev, Continuous-time random walk under time-dependent resetting, *Phys. Rev. E* **96**, 012126 (2017).
- [21] A. S. Bodrova and I. M. Sokolov, Continuous-time random walks under power-law resetting, *Phys. Rev. E* **101**, 062117 (2020).
- [22] M. Montero and J. Villarroel, Monotonic continuous-time random walks with drift and stochastic reset events, *Phys. Rev. E* **87**, 012116 (2013).
- [23] E. W. Montroll and G. H. Weiss, Random walks on lattices. ii, *J. Math. Phys.* **6**, 167 (1965).
- [24] V. M. Kenkre, E. W. Montroll, and M. F. Shlesinger, Generalized master equations for continuous-time random walks, *J. Stat. Phys.* **9**, 45 (1973).
- [25] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* **339**, 1 (2000).
- [26] R. Hilfer and L. Anton, Fractional master equations and fractal time random walks, *Phys. Rev. E* **51**, R848(R) (1995).
- [27] R. Hilfer, Exact solutions for a class of fractal time random walks, *Fractals* **03**, 211 (1995).
- [28] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications* (Springer, Berlin/Heidelberg, 2014).
- [29] T. Sandev and Ž. Tomovski, *Fractional Equations and Models: Theory and Applications* (Springer, Cham, Switzerland, 2019), Vol. 61.
- [30] W. Feller, *An Introduction to Probability Theory and Its Applications*, second edition (Wiley, New York, 1971), Vol. II, pp. xxiv+669.
- [31] Ž. Tomovski, T. K. Pogány, and H. M. Srivastava, Laplace type integral expressions for a certain three-parameter family of generalized mittag-leffler functions with applications involving complete monotonicity, *J. Franklin Inst.* **351**, 5437 (2014).
- [32] A. V. Chechkin, R. Metzler, V. Y. Gonchar, J. Klafter, and L. V. Tanatarov, First passage and arrival time densities for Lévy flights and the failure of the method of images, *J. Phys. A* **36**, L537 (2003).
- [33] V. V. Palyulin, A. V. Chechkin, and R. Metzler, Lévy flights do not always optimize random blind search for sparse targets, *Proc. Natl. Acad. Sci. USA* **111**, 2931 (2014).
- [34] A. M Mathai, R. K. Saxena, and H. J Haubold, *The H-Function: Theory and Applications* (Springer, New York, 2010).
- [35] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, U.K., 2001).
- [36] G. Rangarajan and M. Ding, First passage time distribution for anomalous diffusion, *Phys. Lett. A* **273**, 322 (2000).