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Analytical model for minority games with evolutionary learning

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ABSTRACT

In a recent work [D. Campos, J.E. Llebot, V. Méndez, Theor. Popul. Biol. 74 (2009) 16] we have introduced a biological version of the Evolutionary Minority Game that tries to reproduce the intraspecific competition for limited resources in an ecosystem. In comparison with the complex decision-making mechanisms used in standard Minority Games, only two extremely simple strategies (*juveniles* and *adults*) are accessible to the agents. Complexity is introduced instead through an evolutionary learning rule that allows younger agents to learn taking better decisions. We find that this game shows many of the typical properties found for Evolutionary Minority Games, like self-segregation behavior or the existence of an oscillation phase for a certain range of the parameter values. However, an analytical treatment becomes much easier in our case, taking advantage of the simple strategies considered. Using a model consisting of a simple dynamical system, the phase diagram of the game (which differentiates three phases: *adults crowd*, *juveniles crowd* and *oscillations*) is reproduced.

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1. Minority rules with evolutionary learning

In the last years,the properties of the so-called Minority Games (MGs) have been explored exhaustively [1–3]. Though many different versions of the original game [4] have been proposed, the common element in all of them is the existence of a Minority Rule which is repeated again and again. This rule reads *'a set of agents are offered to choose between different options; after all of them have made their choice those in the less crowded option will be considered the game winners'*. The Minority Rule tries to implement the idea that in systems where the agents compete for limited resources, being in minority is an advantage since you have less competitors to deal with. This idea applies in a very intuitive way, for example, to traffic jams [5,6], where drivers have to predict what is the best route to get point A from point B (or what is the best time to take the car) without prior knowledge of the other drivers' decision. However, most works on MGs have just focused on their applications to financial markets [3], probably due to the availability of accurate data series which facilitates comparison with the results obtained. In that case, the agents are considered as traders that have to choose between selling or buying an option or an asset; so, being in the minority (majority) group will represent a benefit (loss) for them.

Surprisingly, there have been few efforts to bring the MGs into a biological context, albeit the idea of competition for limited resources is of fundamental importance in population ecology. A few exceptions to this are Refs. [7–9], while other works [10,11] have also explored the ecological motivations of these games. We think that this is in part due to the fact that MGs were formulated from the very beginning into a financial context. The first versions of the game were based on the idea that the agents had a pool of many different strategies available, and they decided to choose one or the other by analyzing the history of the game and checking which strategies had performed better for similar situations in the past. Such a complicated decision-making mechanism can be rather appropriate to describe how a trader decides on buying or selling an option, but it is not, for instance, for an animal which has to decide what is the best place to set up its burrow.

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According to these ideas, we proposed recently [12] a new model called the Evolutionary Learning Game (ELG) which implements a Minority Rule to describe competition for resources, but where the decision-making mechanism of individual agents is adapted to a more biological situation. In this model, there are N agents which have to choose repeatedly between two options (A or B) according to the following rules:

(i) One of the two options (in the following, option B) will be the *a priori* best option; this idea is introduced through a Minority Rule with an arbitrary cutoff L [13] (with $0 < L < N/2$). This parameter L would represent the total quantity of resources available for the agents in option A, while agents choosing B have a quantity of $N - L$ resources to share. Hence, if the number of agents choosing A is higher than L the resources *per capita* in option A will be smaller than in option B, and so B will be the winning option. Instead, if the number of agents choosing A is lower than L, the contrary will happen and the *a priori* best option (that with a higher quantity of resources) will not be the winning one.

(ii) At each time step, the winner agents are rewarded with the possibility of reproduction. With probability r each winner will generate a newborn agent. This newborn will replace an existing agent chosen at random, so both winners and losers have the same probability to be eliminated. This process introduces a birth–death mechanism which is absent in standard MGs, where in general the agents go on playing indefinitely as long as they keep winning.

(iii) Decision-making in the ELG is based on very primary mechanisms in order to implement the idea that biological individuals are basically driven by their instinct. Only two strategies are allowed: younger agents (termed as *juveniles*) take their decisions randomly, while *adults* are experienced agents able to predict what is the *a priori* best option, so they will always choose option B.

(iv) The key ingredient in the game is a learning rule that determines when *juveniles* become *adults*. This is governed by a learning probability pp_k characteristic of each individual, where pp stands for phenotypic plasticity (defined biologically as the capacity of an individual to get adapted to its environment). After each time step, the kth individual (if it is a *juvenile*) is given the chance to become *adult* with probability pp_k; if it does, it will behave as an *adult* permanently. So, pp_k is a measure of how fast each individual becomes experienced and is able to take wiser decisions.

(v) Following some experimental evidences from genetics studies [14,15], we will consider that phenotypic plasticity is an heritable trait. So, the values of pp_k will be transmitted from parents to newborns. Accordingly, when the kth agent wins and generates a newborn agent k', the latter will behave initially as a *juvenile* but will be assigned a learning parameter $pp_{k'}$ randomly chosen from an interval of width w centered at pp_k. So, those agents performing best in the game will generate newborns with similar learning capacities to them.

Apart from these rules, we have discussed in Ref. [12] that a realistic ecological model should take into account many other details, e.g. the fact that the number of agents N should vary with time, birth–death mechanisms should be more realistic, etc. These and other possibilities have been explored and will be in further works. The rules above are just those that allow us to reduce the ELG to its simplest form but still keeping its basic features (see the following section).

2. General dynamics of the ELG

As stated above, the strategies available to the agents in the ELG are much simpler than in usual versions of the MG. Instead, complexity is introduced here through a balance between a learning mechanism that leads the agents to choose wiser decisions as their age increases, and a birth–death process that will eliminate even those agents performing best. Despite these differences, the global dynamics of the ELG still keeps much of the properties usually found in Evolutionary MGs [16], as we shall see. In Ref. [12] we already showed that the ELG is able to reproduce the most striking result found in the Evolutionary MG, i.e. the emergence of self-segregated behavior [16,17]. This means that in the stationary situation, reached after multiple iterations of the game, the distribution of pp_k values tends to adopt an U shape (see Fig. 1 in Ref. [12]). So, extreme values $pp_k \to 1$ (learning very fast) or $pp_k \to 0$ (not learning at all) are found to perform best in the game than intermediate pp_k values.

Also, oscillating behaviors are typically found in the game [12,17,18], which reflects that for some periods of time the *a priori* best option (that is, B) is the winning one (especially when the number of adults in the game is small) but it is not for other periods (if the number of adults grows too much, all of them will choose option B and then it will be overcrowded). In Fig. 1(b)–(e) we plot for different values of the parameters L, r and w (see figure caption and legends) the fraction of *juveniles* in the game as a function of time; this shows some of the characteristic oscillations one finds in the ELG. However, oscillations do not arise for the whole range of parameter values, but it is also possible to reach a stationary state where either option B (Fig. 1(a)) or option A (Fig. 1(f)) is always the winning one. As a result, the system tends to a stable situation where the number of juveniles remains approximately constant. In the case of Fig. 1(f), since the *adults* are all losing because option B is overcrowded, we can define this situation as *adults crowd*, while the situation in Fig. 1(a) will be termed in consequence *juveniles crowd*.

To understand in which situations any of the three possible outcomes of the game will arise, we define the average trend of the game by $\langle h \rangle \equiv \frac{1}{T}\sum_{i=1}^{T} \delta(i)$, where $\delta(i)$ is a variable that takes the value '0' if A is the winning option and the value '1' otherwise, and the sum is performed over T consecutive iterations of the ELG. In Fig. 2 we plot $\langle h \rangle$ as a function of the parameter L for $T = 2000$. Interestingly, we observe that the transition from *oscillations* to *juveniles crowd* or to *adults crowd* is not smooth but follows a step-like behavior. This suggests that these three different outcomes could actually be interpreted as three different phases of the game. This result reminds very much that found in Ref. [19] for the Evolutionary

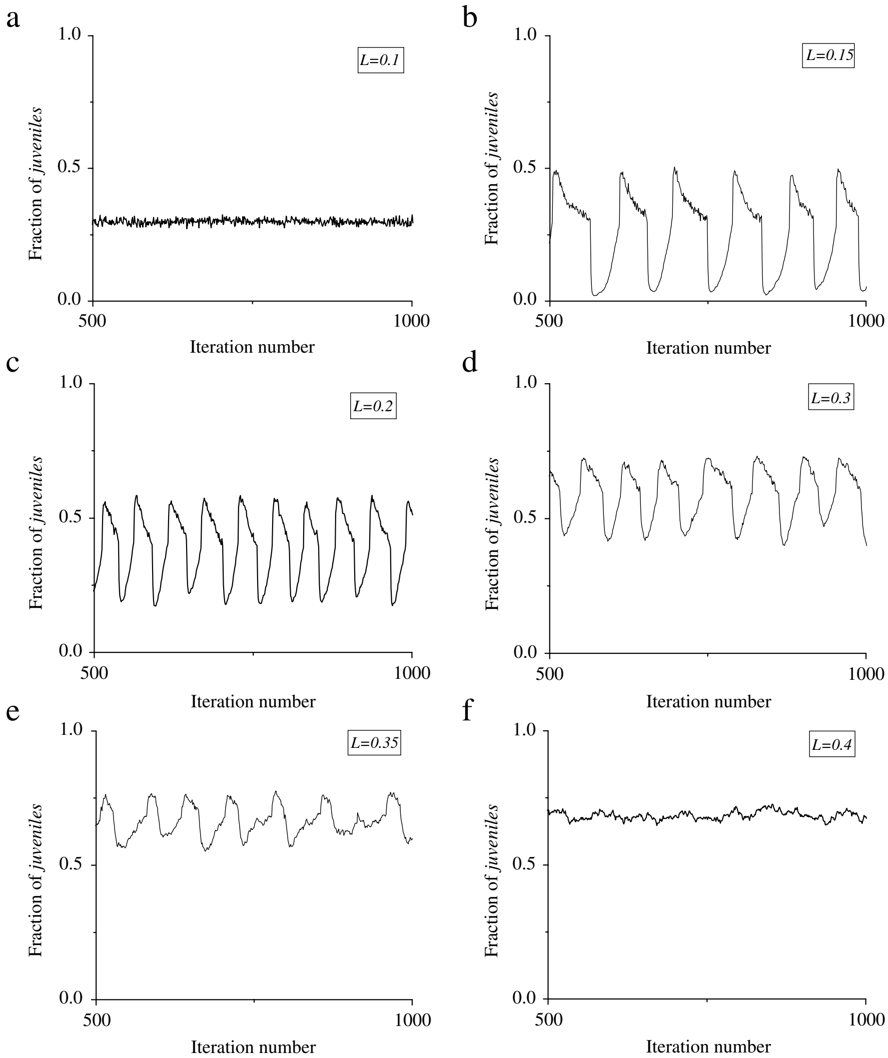

Fig. 1. General dynamics of the ELG. The plots show the fraction of juveniles as a function of time iteration once the stationary situation has far been reached (note that the first time iteration shown is 500). The values of the parameters used are $N = 2001$, $r = 0.3$, $w = 0.15$, while the value of L is changed (see legends).

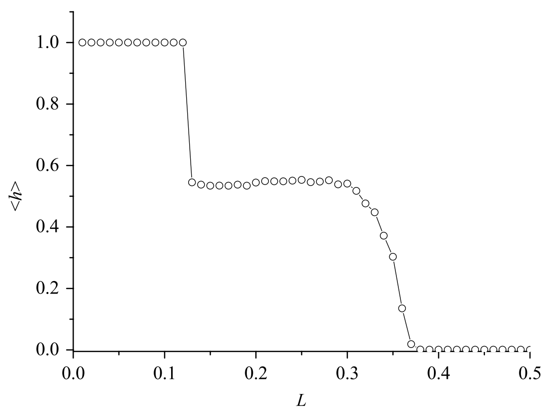

Fig. 2. The average trend $\langle h \rangle$ of the ELG as a function of the parameter L, using the same parameter values as in Fig. 1. $\langle h \rangle = 1$ reflects a situation of *adults crowd*, $\langle h \rangle = 0$ corresponds to *juveniles crowd*, while intermediate values of $\langle h \rangle$ will then correspond to *oscillations*.

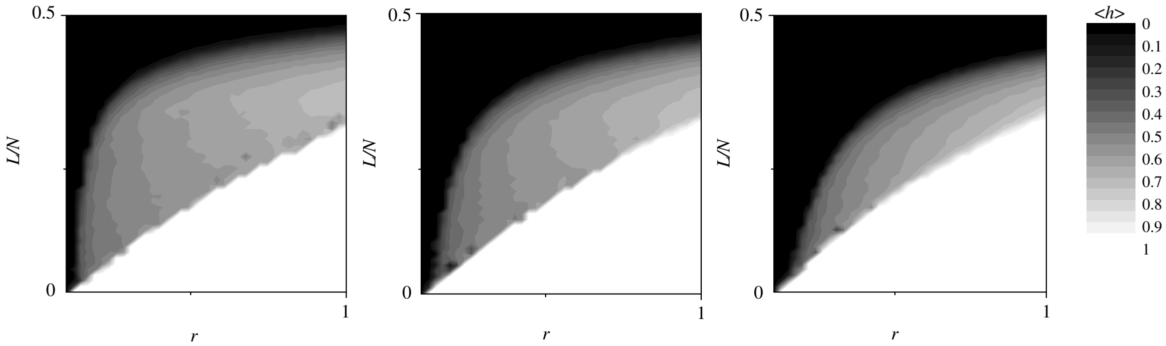

Fig. 3. Phase diagram L/N vs r for the ELG, where the gray scale represents the values of $\langle h \rangle$ (see legend). The three different plots correspond, from the left to the right, to $w = 0.08$, $w = 0.15$ and $w = 0.25$.

MG. There, this step-like behavior has also been observed (see Fig. 2 in Ref. [19]), which is also directly related to the agents distributions analyzed recently in Ref. [20]. Despite these similarities, we stress that a complete analogy between that model and the ELG cannot be established. For example, the frozen states reported in Ref. [19] will not be found in the ELG. In those states, the agents reach a situation where none of them modify their strategies since they do not need it in order to keep on winning in average. That situation is not possible in the ELG, since the birth–death mechanism makes that an evolutionary element is always present in the game.

The phase diagram of the ELG is shown in Fig. 3. There we have plotted again the average trend $\langle h \rangle$ in a gray scale map. Since there are three different parameters in the game, the phase diagram is shown for r versus L, and for different values of w. Fig. 3 confirms the behavior already observed in Fig. 2, that is, the three possible outcomes of the game correspond to regions clearly differentiated. The black zone defines the parameter region for *juveniles crowd* and the white one for *adults crowd*, while the intermediate gray zone corresponds to *oscillations*.

3. Analytical models

In the previous section, we have already checked that the results from the ELG are in consonance with many properties of the Evolutionary MG. However, the strategies followed by the agents to take their decisions are much simpler in our model. One of the main advantages of this is that it facilitates an analytical treatment. For standard MGs, an analytical treatment is possible nowadays, but it has been reached only through extensive research during many years. The framework of the statistical mechanics has been necessary to understand the complex transitions and dynamics observed in those games [2,21,22]. Also, for the Evolutionary MG different analytical approaches (some of them semiempirical) have been proposed [23–25], but they can reproduce only partially the properties of that game. Instead, we shall show in the following that the results of the ELG presented in Section 2 can be reproduced by means of simple dynamic systems.

3.1. Basic model

Let us denote the number of *juveniles* and *adults* at the ith time step by $N_j(i)$ and $N_a(i)$ respectively. The model consists of two equations which accounts for the evolution of these two variables, whose general form will read

$$\begin{aligned} N_j(i+1) &= N_j(i) + b(i) - m(i)N_j(i) - \alpha(i)N_j(i) \\ N_a(i+1) &= N_a(i) - m(i)N_a(i) + \alpha(i)N_j(i) \end{aligned} \tag{1}$$

where $m(i)$ accounts for the mortality probability, $b(i)$ is the number of births, and $\alpha(i)$ is an effective learning parameter that determines the fraction of *juveniles* that become *adults* at the ith time step (so, $\alpha(i)$ can be interpreted as an ensemble average of the pp_k values).

According to the rules of the ELG, half of the *juveniles* are expected in average to choose option A and half of them are expected to choose B, while all the *adults* will choose B always. So that, the condition $N_j(i)/2 < L$ will lead to the option A being the winning one at time step i, with $N_j(i)/2$ being be the number of winning agents. For $N_j(i)/2 > L$ the contrary will happen and the total number of winners will be $N_j(i)/2 + N_a(i)$. In general, the number of births $b(i)$ corresponds to the number of winning agents multiplied by the reproduction factor r. So that, we can write

$$b(i) = \begin{cases} \dfrac{rN_j(i)}{2} & \text{if } N_j(i)/2 < L \\ r\left(\dfrac{N_j(i)}{2} + N_a(i)\right) & \text{if } N_j(i)/2 > L. \end{cases} \tag{2}$$

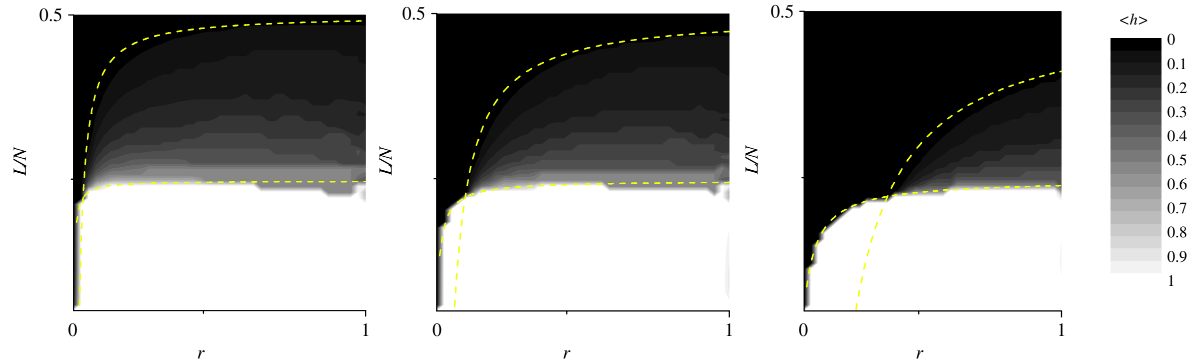

Fig. 4. Phase diagram L/N vs r for the model (4), where the gray scale represents the values of $\langle h \rangle$ (see legend). The three different plots correspond, from the left to the right, to $\alpha = 0.01$, $\alpha = 0.03$ and $\alpha = 0.1$. The dashed lines correspond to the representation of the conditions in (5).

Since each newborn replaces an existing agent, the mortality term $m(i)$ can also be written in a very similar way; the probability that a given agent dies is just $b(i)/N$ so it leads us to

$$m(i) = \begin{cases} \dfrac{rN_j(i)}{2N} & \text{if } N_j(i)/2 < L \\ \dfrac{r}{N}\left(\dfrac{N_j(i)}{2} + N_a(i)\right) & \text{if } N_j(i)/2 > L. \end{cases} \tag{3}$$

Finally, the form of the learning parameter $\alpha(i)$ is more difficult to predict, so for the moment we will consider it a constant (in the next section we will relax this assumption). So that, putting all together the model takes the form

$$\begin{aligned} &N_j(i+1) = N_j(i) + \frac{rN_j(i)}{2}\left(1 - \frac{N_j(i)}{N}\right) - \alpha N_j(i) && \text{if } N_j(i)/2 < L \\ &N_j(i+1) = N_j(i) + r\left(N - \frac{N_j(i)}{2}\right)\left(1 - \frac{N_j(i)}{N}\right) - \alpha N_j(i) && \text{if } N_j(i)/2 > L \end{aligned} \tag{4}$$

where the variable $N_a(i)$ has been eliminated by using the conservation condition $N_j(i) + N_a(i) = N$.

Next, we analyze the steady states N_j^* and their stability. Is is straightforward to find that the possible steady states are

$$\begin{aligned} &N_j(i)/2 < L \rightarrow \begin{cases} N_j^* = 0 \\ N_j^* = N\left(1 - \dfrac{2\alpha}{r}\right) \end{cases} \\ &N_j(i)/2 > L \rightarrow N_j^* = \xi N \end{aligned}$$

where ξ represents the roots of the polynomial $r\xi^2 - (2\alpha + 5r)\xi + 2r = 0$. This polynomial has always one solution in the interval $0 < \xi < 1$ and another one in $\xi > 1$; the latter will be obviously discarded, since it is a non-realistic solution. Also, from a linear stability analysis, it can be proved that the solution $N_j^* = N\left(1 - \frac{2\alpha}{r}\right)$ is stable for $2\alpha < r$; this is exactly the same condition that leads to the instability of the trivial state $N_j^* = 0$. Likewise, the only realistic steady solution for the case $N_j(i)/2 > L$ is found to be always stable.

All this information serves us to understand the basic dynamics of the model (4). For $2\alpha > r$ the system will approach one of the stable solutions available, depending on the initial conditions introduced. If the trivial state $N_j^* = 0$ is reached then we have a situation of *adults crowd*, while for $N_j^* = \xi N$ we will obtain *juveniles crowd*. In the case $2\alpha < r$ the system can reach in principle any of the two non-trivial steady solutions (obtaining again either *adults crowd* or *juveniles crowd*), but also it can happen that the conditions

$$\begin{aligned} &N\left(1 - \frac{2\alpha}{r}\right) > 2L \\ &\xi N < 2L \end{aligned} \tag{5}$$

are satisfied. If this happens, none of the two stable states will ever be reached, since the system jumps from the case $N_j(i)/2 < L$ to $N_j(i)/2 > L$ (and vice versa) before any of these steady values is reached. As a consequence, the conditions in (5) determine the region of parameters where the system keeps jumping from one case to the other, that is, oscillating behavior is obtained.

In Fig. 4 we plot the values of $\langle h \rangle$ obtained numerically from (4) to build the phase diagram using the same gray scale map as in Fig. 3, and for different values of the effective learning parameter α. It can be observed that the similarities between

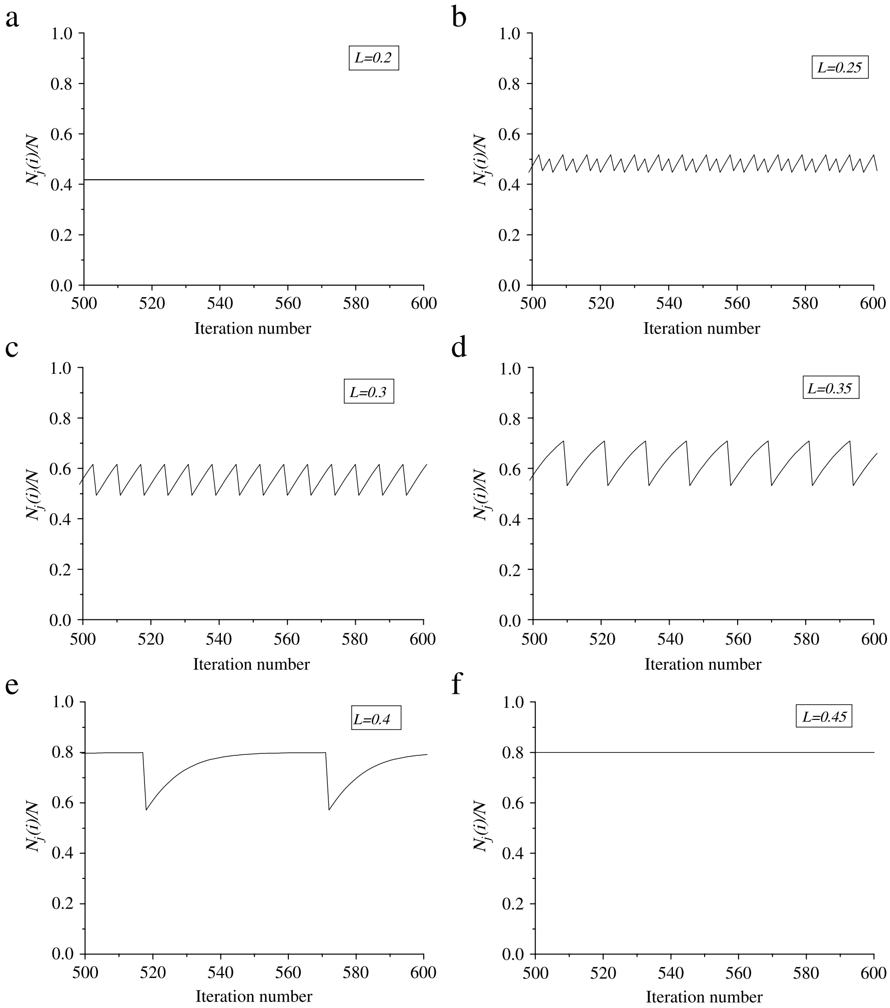

Fig. 5. Fraction of juveniles $N_j(i)/N$ as a function of the iteration number i. The values of the parameters used are $N = 2001$, $r = 0.3$, $\alpha = 0.03$, while the value of L is changed (see legends).

these diagrams and those found from the ELG are clear at a general level. However, the *oscillations* region is not so clearly defined in these diagrams as in Fig. 3, and the specific shape of that region seems to be different, especially for L small. The dashed lines in Fig. 4 represent the parameter conditions (5). As expected, these lines are in agreement with the boundaries of the regions where *oscillations* are observed.

To complete our comparison with the results from the ELG, we show in Fig. 5 the evolution of $N_j(i)/N$ as a function of time for different values of the parameters. It can be checked that, although the model can certainly give rise to either oscillations or stable situations, the shape of the oscillations observed is rather simple compared to the behavior obtained in Fig. 1.

3.2. Model with two learning parameters

Though the model presented in the previous section can certainly capture most of the qualitative aspects of the ELG, the shape of the oscillation region in the phase diagram and the shape of the oscillations themselves does not fit very well the results obtained from the game simulations. These differences are mainly due to the assumption made above that the effective learning parameter α is a constant. Instead, in the ELG it can be observed that the ensemble average of pp_k is far from being a constant when oscillation behavior is obtained (some examples of this are shown in Fig. 6). Trying to take into account the whole dynamics of learning would require to define new functions $N_j(\alpha, i)$ and $N_a(\alpha, i)$ for the number of juveniles and adults with phenotypic plasticity α, so that $N_j(i) = \int_0^1 N_j(\alpha', i)\mathrm{d}\alpha'$. Then Eqs. (1)–(3) should be replaced by

$$\begin{aligned} N_j(\alpha, i+1) &= N_j(\alpha, i) + b(\alpha, i) - m(i)N_j(\alpha, i) - \alpha N_j(\alpha, i) \\ N_a(\alpha, i+1) &= N_a(\alpha, i) - m(i)N_a(\alpha, i) + \alpha N_j(\alpha, i) \end{aligned} \quad (6)$$

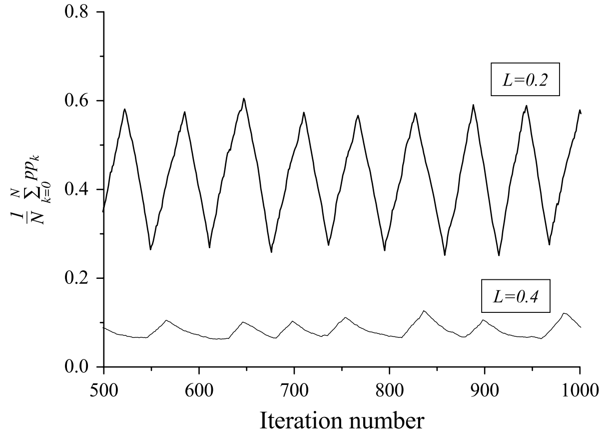

Fig. 6. Ensemble average of the learning parameter pp_k in the ELG as a function of time (iteration number). The parameters used are the same as in Fig. 1, except for the value of L (see legend).

$$b(\alpha, i) = \begin{cases} \displaystyle\int_0^1 W(\alpha \mid \alpha') \frac{N_j(\alpha', i)}{2} \mathrm{d}\alpha' & \text{if } N_j(i)/2 < L \\ \displaystyle\int_0^1 W(\alpha \mid \alpha') \left(N_a(\alpha', i) + \frac{N_j(\alpha', i)}{2} \right) \mathrm{d}\alpha' & \text{if } N_j(i)/2 > L \end{cases} \tag{7}$$

$$m(i) = \begin{cases} \dfrac{rN_j(i)}{2N} & \text{if } N_j(i)/2 < L \\ \dfrac{r}{N}\left(N - \dfrac{N_j(i)}{2} \right) & \text{if } N_j(i)/2 > L \end{cases} \tag{8}$$

where $W(\alpha \mid \alpha')$ represents a transition distribution that determines the probability that an individual with plasticity α' generates a newborn with plasticity α. According to the rule (v) of the ELG which determine how plasticities are inherited (see Section 1) the system of Eqs. (6)–(8) should then be solved for

$$W(\alpha \mid \alpha') = \frac{H(\alpha - \alpha' - w/2) - H(\alpha - \alpha' + w/2)}{2} \tag{9}$$

with $H(\cdot)$ denoting the Heaviside function, and with appropriate boundary conditions at $\alpha = 0, 1$. The corresponding integro-difference system, however, is not analytically manageable due to the nonlinearities involved.

Instead, there is a much simpler generalization we can propose to improve the performance of the model in Section 3.1. It consists of assuming two different populations in the model, each with a different learning parameter, α_1 and α_2. To introduce this into the model, we will also need a new parameter f which tells us what is the probability that an agent with a given learning parameter will generate a newborn with the same learning parameter (so it plays the role of the distribution $W(\alpha \mid \alpha')$).

Introducing these ideas into the derivation presented in the previous section, the equivalent to Eq. (1) would now read

$$\begin{aligned} N_{j1}(i+1) &= N_{j1}(i) + b_1(i) - m(i)N_{j1}(i) - \alpha_1 N_{j1}(i) \\ N_{j2}(i+1) &= N_{j2}(i) + b_2(i) - m(i)N_{j2}(i) - \alpha_2 N_{j2}(i) \\ N_{a1}(i+1) &= N_{a1}(i) - m(i)N_{a1}(i) + \alpha_1 N_{j1}(i) \\ N_{a2}(i+1) &= N_{a2}(i) - m(i)N_{a2}(i) + \alpha_2 N_{j2}(i) \end{aligned} \tag{10}$$

where N_{j1}, N_{j2}, N_{a1}, N_{a2} represent the number of *juveniles* and *adults* with learning parameter α_1 and α_2, respectively. Similarly, $b_1(i)$ and $b_2(i)$ represent the number of births with learning parameter α_1 or α_2.

The form of the parameters $b_1(i)$, $b_2(i)$ is obtained now using analogous arguments to those made for the basic model above. Using $N_j(i) = N_{j1}(i) + N_{j2}(i)$ to denote the total number of *juveniles*, we find

$$b_1(i) = \begin{cases} r\left(f\dfrac{N_{j1}(i)}{2} + (1-f)\dfrac{N_{j2}(i)}{2} \right) & \text{if } N_j(i)/2 < L \\ r\left(f\left(\dfrac{Nj_1(i)}{2} + N_{a1}(i) \right) + (1-f)\left(\dfrac{N_{j2}(i)}{2} + N_{a2}(i) \right) \right) & \text{if } N_j(i)/2 > L \end{cases} \tag{11}$$

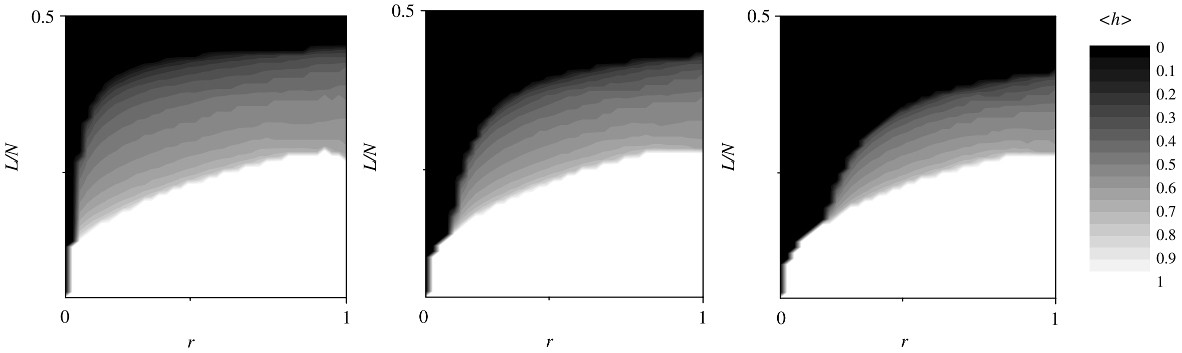

Fig. 7. Phase diagram L/N vs r for the model (13), (14), where the gray scale represents the values of $\langle h \rangle$ (see legend). The three different plots correspond, from the left to the right, to $\alpha_1 = 0.01$, $\alpha_1 = 0.03$ and $\alpha_1 = 0.05$, while the rest of the parameter values used are $N = 2001$, $\alpha_2 = 1$, $f = 0.9$.

$$b_2(i) = \begin{cases} r\left((1-f)\dfrac{N_{j1}(i)}{2} + f\dfrac{N_{j2}(i)}{2}\right) & \text{if } N_j(i)/2 < L \\ r\left((1-f)\left(\dfrac{N_{j1}(i)}{2} + N_{a1}(i)\right) + f\left(\dfrac{N_{j2}(i)}{2} + N_{a2}(i)\right)\right) & \text{if } N_j(i)/2 > L \end{cases} \tag{12}$$

while the mortality $m(i)$ will keep the same form as above in (3). All together, the model with two learning parameters can be expressed by the system of equations

$$\begin{aligned} N_{j1}(i+1) &= N_{j1}(i) + r\left(f\frac{N_{j1}(i)}{2} + (1-f)\frac{N_{j2}(i)}{2} - N_{j1}(i)\frac{N_{j1}(i)+N_{j2}(i)}{2N}\right) - \alpha_1 N_{j1}(i) \\ N_{j2}(i+1) &= N_{j2}(i) + r\left((1-f)\frac{N_{j1}(i)}{2} + f\frac{N_{j2}(i)}{2} - N_{j2}(i)\frac{N_{j1}(i)+N_{j2}(i)}{2N}\right) - \alpha_2 N_{j2}(i) \\ N_{a1}(i+1) &= N_{a1}(i) - \frac{rN_j(i)}{2N}N_{a1}(i) + \alpha_1 N_{j1}(i) \end{aligned} \tag{13}$$

for the case $N_j(i)/2 < L$, and

$$\begin{aligned} N_{j1}(i+1) &= N_{j1}(i) + r\left((1-f)N + (2f-1)N_{a1}(i) + \frac{f-1}{2}N_{j2}(i)\right. \\ &\quad \left. + \left(\frac{3f}{2} - 2 + \frac{N_{j1}(i)+N_{j2}(i)}{2N}\right)N_{j1}(i)\right) - \alpha_1 N_{j1}(i) \\ N_{j2}(i+1) &= N_{j2}(i) + r\left(fN + (1-2f)N_{a1}(i) + \left(1-\frac{3f}{2}\right)N_{j1}(i) - \left(\frac{f}{2} + 1 - \frac{N_{j1}(i)+N_{j2}(i)}{2N}\right)N_{j2}(i)\right) - \alpha_2 N_{j2}(i) \\ N_{a1}(i+1) &= N_{a1}(i) - r\left(1 - \frac{N_{j1}(i)+N_{j2}(i)}{2N}\right)N_{a1}(i) + \alpha_1 N_{j1}(i) \end{aligned} \tag{14}$$

for $N_j(i)/2 > L$. Again, the conservation condition $N = N_{j1}(i) + N_{j2}(i) + N_{a1}(i) + N_{a2}(i)$ has been used to eliminate the variable $N_{a2}(i)$.

The complete analysis of the steady solutions and the stability of the system (13), (14) is extremely lengthy so one rather needs the help of some software to deal with it. Hence, we cannot reproduce the whole analysis here. It is enough to say that one can find that the model (13), (14) behaves qualitatively in the same way as the basic model (4). If the condition

$$\alpha_1 > \frac{r^2(1-2f) + 2\alpha_2 rf}{2(2\alpha_2 - rf)} \tag{15}$$

is fulfilled, then the system either tends to the trivial steady solution $\left(N_{j1}^*, N_{j2}^*, N_{a1}^*\right) = \left(0, 0, \frac{\alpha_1}{\alpha_1+\alpha_2}N\right)$ or to a non-trivial state obtained from the case $N_j(i)/2 > L$. The former reflects a situation of *adults crowd*, while the latter corresponds to *juveniles crowd*. Instead, if condition (15) is not satisfied the system will either reach one of the two possible non-trivial states available (one from $N_j(i)/2 < L$ and one from $N_j(i)/2 > L$) or give rise to *oscillations*. The specific necessary conditions for oscillating behavior, however, cannot be made explicit as in (5).

The phase diagrams one obtains for the model with two learning parameters are shown in Fig. 7 for different values of α_1, α_2 and f. Compared to the results from the basic model (Fig. 4) it is clear that this model fits much better the *oscillations* region obtained from the ELG. Also, the shape of the oscillations (Fig. 8) is found to exhibit a much similar behavior to those

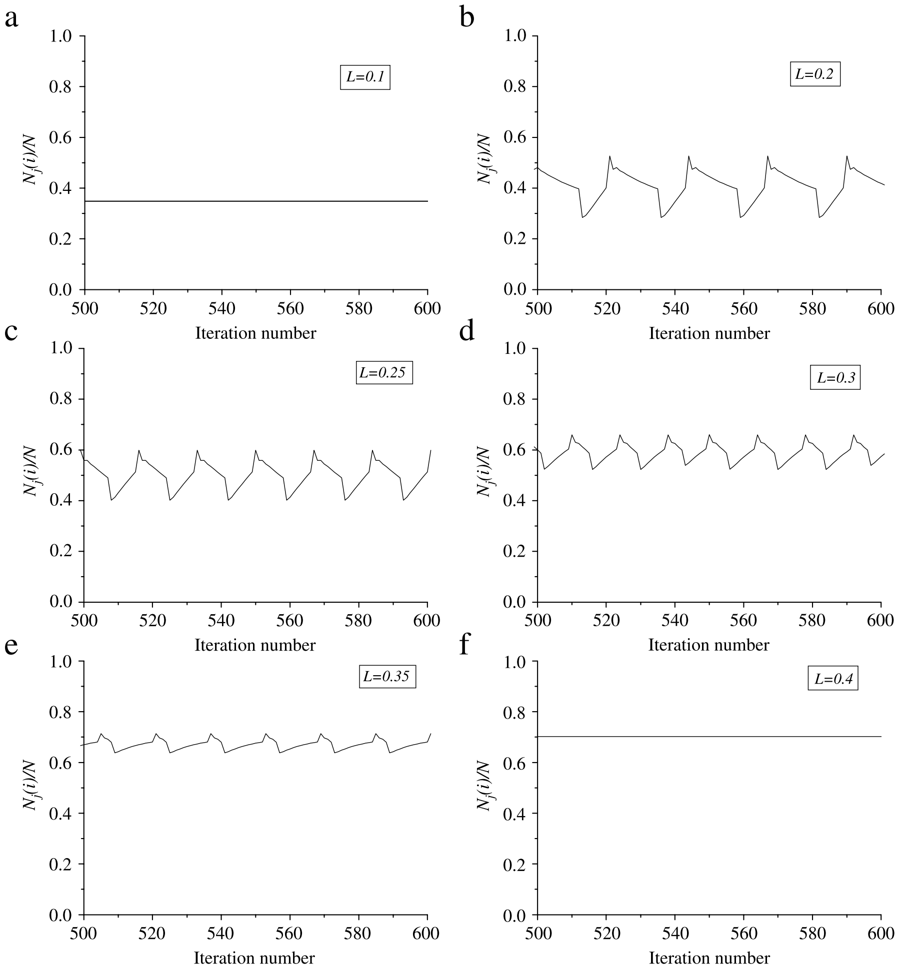

Fig. 8. Fraction of juveniles $N_j(i)/N$ as a function of the iteration number i. The values of the parameters used are the same as in Fig. 7 with $\alpha_1 = 0.03$, while the value of L is changed (see legends).

reported in Fig. 1. So that, the weaknesses detected for the basic model in Section 3.1 are strongly corrected by means of the generalized version proposed. However, it must be stressed that the agreement cannot be considered perfect at all, since the model with two learning parameters represents still a toy approximation to the ELG. Obviously, considering three, four... learning parameters would improve the agreement. However, this is clearly out of the scope of the present work and it is not even necessary, since in most cases one is just interested in understanding the qualitative properties of these agent models.

4. Discussion: ELG versus MGs

After checking that many of the properties of the ELG can be easily reproduced by such simple models, one may immediately wonder why the same cannot be done for standard MGs. What are then the specific elements which have been removed from the standard model? We have already mentioned throughout the text that this is due to the extremely simple decision-making rules considered, but now we will elaborate on this idea. Basically, the ELG is much simpler because, compared to standard MGs, the *number of degrees of freedom* in the game has been drastically reduced and *memory effects* are absent. Note that, although the Minority Rule is the common feature in all these games, the levels of complexity found in standard MGs are a consequence of many other elements there:

- First, the strategies assigned to each individual are different in those games, which makes individuals different or even unique (in case the number of strategies available is of the same order or larger than the number of individuals). If the kth individual has a set of S strategies available, it will choose the one with a higher valuation, where the specific values p_{kj} ($j = 1, 2, \ldots, S$) of these valuations are updated according to the performance of each strategy during the game. If $S = 2$, then the sign of the function $q_k \equiv p_{k_1} - p_{k2}$ determines which is the strategy chosen by the kth individual. So, we need to know q_k for all the N individuals to determine the outcome of the game. At the end, these q_k represent the

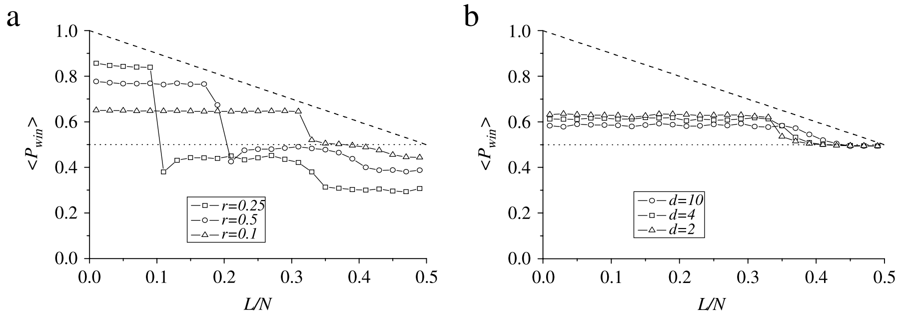

Fig. 9. Efficiency levels for the ELG (left) and the Evolutionary MG (right) as a function of L/N for different values of r (ELG) and d (Evolutionary MG). In both cases the values $w = 0.15$, $N = 2001$ have been used. In the Evolutionary MG we have taken $M = 4$ for the memory parameter and 1 for the prize-to-fine ratio.

degrees of freedom of the game and define a phase space where the formalism of statistical mechanics applies [2]. In the evolutionary version of MGs something similar happens, since each individual is assigned a valuation p_k that measures its performance during the game [16]. On the contrary, in the ELG we just have two different behaviors or subpopulations (*adults* and *juveniles*) whose behavior can be predicted (except for the stochastic fluctuations). So that, we just need to know the number of *juveniles*, which is the only degree of freedom, to determine the outcome in the game.

- Second, the strategies used by the individuals are based on the past outcomes, which introduces a memory effect. Instead, the ELG described above is clearly a Markovian model (as are the analytical approximations proposed in Section 3). Note that memory is a basic feature of standard MGs; an individual playing in a memoryless version of the original game would choose always the same option A or B repeatedly. The Evolutionary MG, instead, do admit a non-trivial memoryless version which has been analyzed in Ref. [26].

Also, note that the Evolutionary MG introduces a stochastic component in the model. Stochasticity, combined with the role of the Minority Rule, introduces some sort of complexity too. This level of complexity, however, is also present in the ELG -note that the evolutionary learning rule used in the ELG is inspired on a similar rule used in the Evolutionary MG [16].

Do then these strong simplifications make the ELG an uninteresting model? We do not think so. We stress that the phenomenology found in our model (summarized in Figs. 1–3 and the results reported in Ref. [12]) reproduces qualitatively most of the characteristics of the Evolutionary MG. This suggests that the effects of stochasticity present in both models, together with the Minority Rule, dominate over the effects due to memory or those due to the decision-making details. So, inductive reasoning and other key concepts associated to decision mechanisms in MGs does not seem to change the qualitative behavior of the model in an evolutionary scenario, while it is expected that the quantitative differences are more important. In order to evaluate this idea we can compare the average efficiency (which is a basic parameter in MGs) of the ELG (primitive decision-making) and the Evolutionary MG (sophisticated decision-making). It does not exist an univocal way to define efficiency within this context; however, an appropriate definition for our purposes can be the time-averaged winning probability of the agents [17], which we denote by $\langle P_{win} \rangle$. This is a direct measure of how fruitfully individuals use the limited resources available in the system.

In Fig. 9 we plot the values found for the two games. For the Evolutionary MG (Fig. 9(b)) we have considered a memory parameter $M = 4$, a prize-to fine ratio equal to 1 and different lower thresholds d leading to the removal of an agent (see Ref. [16] for details). We can observe that efficiency levels in the ELG (Fig. 9(a)) are in general comparable to those in the Evolutionary MG, despite primary decision-making mechanisms. To facilitate the comparison, we also plot in Fig. 9(a) and (b) the efficiency corresponding to a system where individuals behave randomly ($\langle P_{win} \rangle = 1/2$, dotted line) and the case of optimum efficiency ($\langle P_{win} \rangle = 1 - L/N$, dotted lines). We can observe that primary behavior of individuals in the ELG make this system less efficient than the random system for L large, specially if r is small (for r small very few newborns appear so the number of adults grows too much and they perform worse in consequence). A possible interpretation of these results is that the ELG cannot be expected to represent a realistic situation in systems where the two possible options A or B have similar resources available (that is, for L large) since in that case random decision-making is more efficient. On the contrary, in situations of strong asymmetry between the two options A and B, the evolutionary learning rule defined in the ELG seems to perform quite well and leads to quite high efficiency levels, even above those found for the Evolutionary MG.

5. Conclusions

In summary, we have shown that the ELG, which was presented for the first time in Ref. [12], is not just able to reproduce many of the properties of standard models based on a Minority Rule, but admits an easy analytical approximation in terms of dynamical systems. So, although the original idea of the model was just to obtain a more biological version of MGs, the availability of a simple analytical treatment can make the ELG or similar versions an attractive choice for the modelization of many complex systems. Here, two analytical approximations to the ELG have been presented. The first one, which reproduces reasonably the results from the ELG at a qualitative level, can be reduced just to a single-species model, so a complete analytical treatment results very easy. If a better agreement with the game dynamics wants to be achieved, a slightly more sophisticated approximation (Section 3.2) can be used, at the price of turning the analytical resolution of the model really cumbersome.

We stress that we still know relatively few about our model if compared with the extensive literature on standard MGs. So that, we think that the future works should be focused on exploring precisely the similarities and differences between our model and standard MGs. For this purpose, probably it would be convenient to carry out statistical mechanics approaches similar to those for the MGs mentioned throughout the text. Also, we hope that the potential biological applications of the model can be exploited in further works. This will require (i) to identify those real systems in nature where extreme environmental constraints can lead to limited resource conditions as those required for a Minority Rule to hold, and (ii) to explore more realistic and sophisticated versions of the model (specifically, regarding the birth–death mechanisms).

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