Phase transitions in optimal search times: How random walkers should combine resetting and flight scales

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(Received 19 August 2015; published 9 December 2015)

Recent works have explored the properties of Lévy flights with resetting in one-dimensional domains and have reported the existence of phase transitions in the phase space of parameters which minimizes the mean first passage time (MFPT) through the origin [L. Kusmierz *et al.*, Phys. Rev. Lett. **113**, 220602 (2014)]. Here, we show how actually an interesting dynamics, including also phase transitions for the minimization of the MFPT, can also be obtained without invoking the use of Lévy statistics but for the simpler case of random walks with exponentially distributed flights of constant speed. We explore this dynamics both in the case of finite and infinite domains, and for different implementations of the resetting mechanism to show that different ways to introduce resetting consistently lead to a quite similar dynamics. The use of exponential flights has the strong advantage that exact solutions can be obtained easily for the MFPT through the origin, so a complete analytical characterization of the system dynamics can be provided. Furthermore, we discuss in detail how the phase transitions observed in random walks with resetting are closely related to several ideas recurrently used in the field of random search theory, in particular, to other mechanisms proposed to understand random search in space as mortal random walks or multiscale random walks. As a whole, we corroborate that one of the essential ingredients behind MFPT minimization lies in the combination of multiple movement scales (regardless of their specific origin).

DOI: 10.1103/PhysRevE.92.062115 PACS number(s): 02.50.−r, 87.10.−e, 05.40.Fb

I. INTRODUCTION

Resetting can be defined as the interruption of a process in such a way that it is brought back instantaneously from its present state to another fixed state (presumably the initial one) and allowed to evolve once again from there. In the last years there has been a growing interest for studying classical and well-known stochastic transport processes (e.g., Brownian motion, continuous-time random walks, etc.) when subject to stochastic resetting [1–5]. This idea of allowing the particles to start anew its process after some random time has been claimed to have potential applicability to the study of many different systems. For instance, animal foraging is often constrained by the existence of a homing dynamics that makes individuals return from time to time to its nest [6,7], leading to the ecological concept of *central place foraging*. Alternatively, the tendency to revisit well-known places in animals can be more complex and be driven by sophisticated memory mechanisms [8,9], a fact which is of particular interest since similar ideas have been used to explain human mobility patterns, too [10,11]. These behavioral mechanisms, driven by higher cognitive processes, require an extension of the concept of resetting whose properties have just begun being understood [12,13]. In a different context, the transport of information packets through both wired or wireless networks is often subject to transmission losses (errors, overflow, ...), which may require the transmission of new packets after some time (a mechanism which also coincides with resetting under proper assumptions) for increasing the probability of reaching a target whose location is unknown [14]. Similar mechanisms could apply too to beam propagation through optical fiber with random inhomogeneities [3] and many other transport processes on graphs and complex media.

Beyond this practical interest, processes with resetting are attractive from a theoretical perspective as they can induce dramatic changes in the system dynamics if compared to their resetting-absent counterparts. So, resetting facilitates the emergence of nonequilibrium stationary states in different scenarios as in free diffusion [5,15], in motion subject to potential landscapes [16], or in coagulation-diffusion processes [17]. Also, it can induce heavy-tailed decays in the probability distribution of models of transport and/or growth [1,18]. On more mathematical grounds, resetting processes have been found to have interesting connections to some variants of the *multiarmed bandit* problem [19,20], which may contribute to extend their potential range of interest.

The focus of this work is in the use of stochastic resetting as a possible mechanism for optimizing random spatial searches. The relevance of this problem has been already recognized and discussed in some of the aforementioned works (see, e.g., [1,2,4]), but a more specific discussion, addressing the connections of resetting to well-known concepts and results from the literature on random search theory, is still lacking. We aim thus at covering this existing gap.

First of all, let us note that search efficiency can be measured in many different ways attending to time, energetic, or other biological or physical constraints [21]. In agreement with most statistical physics approaches to the field we identify here efficiency with the time required to reach the target, so we will focus on the mean first passage time (MFPT) of the random walker through the target location as the main magnitude of interest. Previous works have already determined the existence of an optimal value of the resetting rate r which minimizes the MFPT of Brownian walkers through the origin in a semi-infinite media [1,2]. The extension of this idea to the case of particles moving according to Lévy flights has revealed the existence of a first-order transition in the optimal values of the phase space (r,μ) (with μ the Lévy index) for which the MFPT becomes a minimum [4]. So, for small values of the initial position x_0, intermediate optimal values $(r^*,\mu^*) = (0.25,1.2893\ldots)$ are found, independent of the specific value of x_0. This regime extends up to a critical value $(x_0)_{cr}$ such that for $x_0 > (x_0)_{cr}$ the optimal combination becomes $(0.22145\ldots,0)$.

Here, we will revisit this problem and will show how actually first-order phase transitions (although of a slightly different nature) do also appear also for simpler random walks, in particular for walks with exponentially distributed flights. For this purpose, we will present two different resetting mechanisms (Sec. II) and will compute their MFPT for the case of motion in the positive semiaxis with a target located at the origin (Sec. III), and then for the case of a finite domain with periodic boundary conditions (Sec. IV). Furthermore, we will try to establish a connection between these results and other models as mortal random walks and multiscale random walks which have been used previously in the random search literature. As a whole, we will show the existence of a common force behind all these motion mechanisms driving MFPT minimization. This force is strongly related to the compromise between efficiently covering nearby regions (*exploitation*) and the *exploration* of new areas, as happens in Lévy or multiscale search strategies [22–24]. From this perspective, resetting must be viewed as a mechanism that promotes *exploitation* as it allows the possibility to revisit regions which may have been initially missed. To clarify the reach of this idea, we will also explore (Sec. V) the case of delayed resetting, which allows us to understand how the MFPT dynamics gets modified if resetting is not an instantaneous process but takes a finite time to occur (an idea which also provides an additional level of realism for animal foraging and other search applications). We will finish in Sec. VI by summarizing the main conclusions from the paper.

II. FIRST PASSAGE PROPERTIES OF RANDOM WALKS WITH RESETTING

We will consider for the moment random walkers moving in a one-dimensional infinite domain (which under proper assumptions provide a convenient representation of many realistic search processes [24,25]), and will focus on the role that stochastic resetting plays on the MFPT through the origin, given the walker starts from an arbitrary initial position x_0. Note in advance that resetting is not expected to be a convenient mechanism for searching unless the target is relatively close to x_0; otherwise, resetting would just prevent the walker from reaching new unexplored areas. So, the use of an infinite domain with resetting can effectively serve to describe the search for nearby targets in finite domains provided that the domain size is much larger than the typical distance covered before resetting.

In contrast with the cases studied in previous works, we will rather consider here that walkers move according to an isotropic velocity model. So, the flights will not be done instantaneously after some waiting time but the transition occurs progressively at a fixed velocity $+v_0$ or $-v_0$, each with probability $\frac{1}{2}$. We do so because we consider that a velocity model is probably a more realistic choice for most search applications; anyway, we stress that the extension of our results to the case of instantaneous jumps separated by waiting times would be straightforward.

We will compare the results for two different resetting mechanisms for the sake of completeness. The first one consists of giving the walker the possibility to reset its position to x_0 whenever a single flight is completed. Accordingly, we will denote by $X_1, X_2, \ldots$ the successive positions of the particle after the first, second, ... event (where each event can be either a flight or a reset). Thus, the position of the particle after the $(i+1)$th event is chosen according to the rule

$$X_{i+1} = \begin{cases} x_0, & \text{with prob. } r \text{ if the } i\text{th event was not a reset} \\ X_i \pm v_0 U_i, & \text{otherwise.} \end{cases} \tag{1}$$

Here, the flight durations U_i's are independent and positive random times determined by the probability distribution function (PDF) $\varphi(t)$, and the symbol $\pm$ denotes that the increment in X_i can be positive or negative with the same probability. In the following, we will term this resetting mechanism as being *subordinated to flights* since the statistics of the flights determines in part the rate at which resetting will occur. Note that this mechanism is slightly different to that proposed in [4] since here two consecutive reset events are not allowed, but they must be necessarily separated by at least one flight. This has the advantage that the dynamics in the limit $r \to 1$ will be nontrivial since there the walker will successively carry out one-flight excursions separated by reset events. Instead, if consecutive resets were allowed as in [4], then for $r \to 1$ the particle would be kept permanently at x_0 (which looks somewhat unrealistic) and the MFPT would diverge in that limit.

Alternatively, we will compare this with a second mechanism (which has been already used in [3]) in which the statistics of resetting is independent of jumps. In this case, flight times are again distributed according to $\varphi(t)$, but the successive times at which a reset occurs are also i.i.d. variables which follow its own PDF $\theta(t)$ [independent from $\varphi(t)$]. So, in this case resetting is not necessarily driven by a single probability r but could be allowed to follow a much more complex dynamics. By introducing this second mechanism (termed as *resetting independent of motion*) we will be able to check if subordination to the motion process has any influence on how resetting affects the values of the MFPT and the dynamics of optimal search.

So, the evolution of our random walkers will be completely determined for a particular choice of v_0 and $\varphi(t)$ [also $\theta(t)$, in the case of *motion-independent resetting*]. To compute the MFPT, we will determine first the rate $q(t;x_0,x_0^*)$ at which crossings through the origin occur at time t for a free random walk starting from x_0. The parameter x_0^* gives the position of the walker immediately after it is reset (for the moment we will set x_0^* as independent of x_0 for convenience, while in the end we are mainly interested in the case $x_0^* = x_0$). Then, we can apply the renewal property [7,26,27]

$$q(t;x_0,x_0^*) = f(t;x_0,x_0^*) + \int_0^t dt' f(t';x_0,x_0^*) q(t-t';0,x_0^*), \tag{2}$$

where $f(t;x_0)$ is the corresponding first passage distribution through the origin. So, trajectories contributing to $q(t;x_0,x_0^*)$ are divided into those which cross the origin at t for the first time (first term on the right-hand side) and those which did it for the first time at a previous time t' (last term on the right-hand side). The connection of the rate function to the free propagator $P(x,t;x_0,x_0^*)$ (i.e., the probability density of particles in the absence of targets and resetting) for the case of velocity models with constant speed, is simply given by [28]

$$q(t;x_0,x_0^*) = \begin{cases} v_0 P(0,t;x_0,x_0^*), & \text{if } x_0 \neq 0 \\ v_0 P(0,t;x_0=0,x_0^*) - \frac{1}{2}\delta(t), & \text{if } x_0 = 0. \end{cases} \tag{3}$$

Here, the additional term $-\frac{1}{2}\delta(t)$ has been introduced to neglect crossings at $t=0$ by explicitly imposing the condition $q(0;0,x_0^*)=0$. This choice is done because we consider that for $x_0=0$ the MFPT value should include only *returns* to the target for some $t>0$; stopping the process at $t=0$ would not fit, in our opinion, the intuitive meaning of what a *search* process is.

The MFPT can be obtained now easily by using for instance Laplace transform techniques [25,29]. So, if $g(s) = \int_0^\infty e^{-st} g(t)dt$ represents the Laplace transform of an arbitrary time-dependent function $g(t)$, then one obtains from (2) the expression for the MFPT

$$\begin{aligned}\langle T\rangle &= -\lim_{s\to 0}\frac{df(s;x_0,x_0^*)}{ds} = -\lim_{s\to 0}\frac{d}{ds}\left[\frac{q(s;x_0,x_0^*)}{1+q(s;x_0=0,x_0^*)}\right] \\ &= -\lim_{s\to 0}\frac{d}{ds}\left[\frac{v_0P(x=0,s;x_0,x_0^*)}{1/2+v_0P(x=0,s;x_0=0,x_0^*)}\right],\end{aligned} \tag{4}$$

where the first identity follows from the relation $\lim_{s\to 0}\frac{df(s;x_0,x_0^*)}{ds} = \lim_{s\to 0}\int_0^\infty dt\, e^{-st} t f(t;x_0,x_0^*)$. Also, note that Eq. (4) implicitly contains the expression for the first passage distribution $f(s;x_0,x_0^*) = q(s;x_0,x_0^*)/[1+q(s;0,x_0^*)]$ in the Laplace space.

Note that renewal properties like that in (2) are strictly valid only for Markovian processes (although they can often be used as a reasonably good approximation for more sophisticated cases, especially regarding MFPT computation [29]). So, in the following we will focus for simplicity on situations which either are Markovian or, equivalently, admit a Markovian embedding.

A. Resetting subordinated to motion

The velocity model driven by the process (1) can be conveniently described through a continuous-time random walk (CTRW) scheme [3,25]. For this we introduce the probability density $j(x,t;x_0,x_0^*)$ for the particles starting a flight from x at time t. This allows us to write the mesoscopic balance equation

$$\begin{aligned} j(x,t;x_0,x_0^*) &= \delta(t)\delta(x-x_0) + (1-r)\int_0^t dt'\int_{-\infty}^{\infty} dx'\Psi(x',t') \\ &\quad\times j(x-x',t-t';x_0,x_0^*) + r\delta(x-x_0^*) \\ &\quad\times\int_0^t dt'\int_{-\infty}^{\infty} dx\,\varphi(t')j(x,t-t';x_0,x_0^*). \end{aligned} \tag{5}$$

The second and third terms on the right-hand side of (5) account for flights starting after another flight or after a reset, respectively. This, together with the initial condition term $\delta(t)\delta(x-x_0)$, prevents the particles from being reset until having completed at least its first flight. In the expression (5) we have introduced the joint probability $\Psi(x,t)$ of performing a jump of length x with constant speed v_0 during time t. Using that flights are done at speed v_0 or $-v_0$, each with the same probability, these PDFs can be easily related to the PDF of flight times through

$$\begin{aligned}\Psi(x,t) &= \frac{1}{2}[\delta(x-v_0t)+\delta(x+v_0t)]\varphi(t) \\ &= \frac{1}{2v_0}\delta\left(t-\frac{|x|}{v_0}\right)\varphi(t).\end{aligned} \tag{6}$$

The free propagator for this scheme can be defined as the probability density of particles that are located at point x at time t, i.e.,

$$\begin{aligned} P(x,t;x_0,x_0^*) &= \int_0^t dt'\int_{-\infty}^{\infty} dx'\phi(x',t')j(x-x',t-t';x_0,x_0^*),\end{aligned} \tag{7}$$

where we have defined

$$\phi(x,t) \equiv \frac{1}{2}[\delta(x-v_0t)+\delta(x+v_0t)]\int_t^\infty dt'\varphi(t'). \tag{8}$$

This last function gives us the probability that a single flight has not finished yet after having traveled during a time t and having covered (either to left or right) a distance v_0t. To find an explicit expression for the free propagator we introduce the Fourier-Laplace transform of an arbitrary function $g(x,t)$ as

$$g(k,s) = \int_0^\infty dt\, e^{-st}\int_{-\infty}^{\infty} dx\, e^{-ikx} g(x,t). \tag{9}$$

In the Fourier-Laplace space, the expressions (5)–(8) become simplified by virtue of the space and time convolution theorems. For example, transforming by Fourier-Laplace equation (5) we obtain

$$\begin{aligned}\hat{j}(k,s;x_0,x_0^*) &= e^{-ikx_0} + (1-r)\hat{\Psi}(k,s)\hat{j}(k,s;x_0,x_0^*) \\ &\quad + re^{-ikx_0^*}\hat{j}(k=0,s;x_0,x_0^*)\tilde{\varphi}(s),\end{aligned} \tag{10}$$

where the hat and tilde symbols mean Fourier-Laplace and Laplace transforms, respectively, and k and s are the corresponding Fourier and Laplace arguments. Setting $k=0$ into Eq. (10) we can solve for $\hat{j}(k=0,s;x_0,x_0^*)$, which yields

$$\hat{j}(k=0,s;x_0,x_0^*) = \frac{1}{1-\tilde{\varphi}(s)}, \tag{11}$$

and then introduce this again into (10) to find a closed expression for $\hat{j}(k,s;x_0,x_0^*)$. Finally, by transforming Eq. (7) by Fourier-Laplace we obtain $\hat{P}(k,s;x_0,x_0^*) = \hat{\phi}(k,s)\hat{j}(k,s;x_0,x_0^*)$. This, combined with Eq. (10), yields a closed expression for $P(k,s;x_0,x_0^*)$ as

$$\begin{aligned}\hat{P}(k,s;x_0,x_0^*) &= \frac{\hat{\phi}(k,s)}{1-(1-r)\hat{\Psi}(k,s)}\left[e^{-ikx_0} + \frac{r\tilde{\varphi}(s)}{1-\tilde{\varphi}(s)}e^{-ikx_0^*}\right],\end{aligned} \tag{12}$$

where $\hat{\Psi}(k,s)$ and $\hat{\phi}(k,s)$ are given by

$$\hat{\Psi}(k,s) = \tfrac{1}{2}[\tilde{\varphi}(s-ikv_0)+\tilde{\varphi}(s+ikv_0)] \tag{13}$$

and

$$\hat{\phi}(k,s) = \tfrac{1}{2}[\tilde{\varphi}^*(s-ikv_0)+\tilde{\varphi}^*(s+ikv_0)]. \tag{14}$$

While the system (12)–(14) can be formally written for a wide range of flight distributions $\varphi(t)$, inverting it from Fourier space to real space [which is necessary to apply (3)] is extremely cumbersome except for some particular choices. So, we will focus now for simplicity in the case where the dynamics of flights is Markovian. This corresponds to the case when the flight time distribution is exponential, i.e., $\varphi(t) = \lambda e^{-\lambda t}$. Replacing this expression into Eq. (12), we find after some algebra

$$\hat{P}(k,s;x_0,x_0^*) = \frac{(s+\lambda)e^{-ikx_0} + r\lambda\left(1+\frac{\lambda}{s}\right)e^{-ikx_0^*}}{(s+\lambda)(s+r\lambda)+v_0^2k^2}, \qquad (15)$$

and after inverting the Fourier transform

$$\tilde{P}(x,s;x_0,x_0^*) = \frac{1}{2v_0}\sqrt{\frac{s+\lambda}{s+r\lambda}}e^{-\sqrt{(s+\lambda)(s+r\lambda)}|x-x_0|/v_0} + \frac{r\lambda}{2sv_0}\sqrt{\frac{s+\lambda}{s+r\lambda}}e^{-\sqrt{(s+\lambda)(s+r\lambda)}|x-x_0^*|/v_0}. \qquad (16)$$

Inserting this expression into (3) and (4) we will finally obtain the expression for the first passage time distribution and the MFPT. For simplicity, we just show the result for the case of interest $x_0^* = x_0$, which reads as

$$f(s:x_0,x_0^*) = \frac{s+r\lambda}{r\lambda + s\left(1+\sqrt{\frac{s+r\lambda}{s+\lambda}}\right)e^{\sqrt{(s+\lambda)(s+r\lambda)}|x_0|/v_0}} \qquad (17)$$

and

$$\langle T\rangle = \frac{1}{\lambda r}\left(\frac{1+\sqrt{r}}{e^{-\lambda\sqrt{r}|x_0|/v_0}} - 1\right). \qquad (18)$$

This result will be discussed in greater detail in Sec. III. For the moment, let us note that by comparing this with the results in [4] for the case of Lévy flights, we observe that the scaling $\lim_{r\to 0}\langle T\rangle \sim r^{-1/2}$ is kept. However, the universal result for $x_0 \to 0$ reported there [see Eq. (8) in Ref. [4]], i.e., $\lim_{x_0\to 0}\langle T\rangle = (\sqrt{r}-r)^{-1}$, is not recovered. This is a consequence of the rule (1) we have imposed; by forcing the walker to make at least one flight before being reset, we have introduced some sort of memory in the system (in the sense that not all events are equivalent). As stated by the authors in [4], their result is a consequence of the Sparre-Andersen theorem, according to which there is a universal asymptotic decay $\sim t^{3/2}$ for the PDF of return times to the origin in isotropic and memoryless random walks in one dimension. In consequence, the necessary conditions for this theorem to hold are not met in our case. Actually, the differences between the dynamics in the two cases are clear by noting that the expression $(\sqrt{r}-r)^{-1}$ leads to the divergence of the MFPT as $r \to 1$, as we have mentioned above. Instead, it is easy to see from (18) that the MFPT for our case does not diverge in that limit.

B. Resetting independent of motion

If the resetting mechanism follows its own time dynamics [according to the time PDF $\theta(t)$, as described above], then it is not convenient to use a continuous-time random walk scheme as done in the previous section to derive the free propagator of the walk. Instead, we will use the analogy between random walks with resetting and *mortal* random walks. Following the nomenclature of several recent works [28,30,31], *mortal* random walks are defined as walks subject to a mortality function such that after a random time, governed by a given PDF, the particle will disappear (that is, it will "die"). So, if we define in our random walks with resetting an excursion as the action that goes from $t = 0$ to the first reset event or, alternatively, from the ith to the $(i+1)$th reset event, then every one of these excursions can be interpreted as an independent mortal random walk. Consequently, the random walk with resetting consists of the time convolution of successive mortal random walks governed by the "mortality" function $\theta(t)$.

We will focus again for simplicity in the Markovian case, so we choose for the resetting mechanism $\theta(t) = \omega_m e^{-\omega_m t}$, with ω_m being the frequency of resets (or, alternatively, the "mortality" rate). So, note that we introduce a change in notation if we compare to the case of *resetting subordinated to motion* since the role of the resetting probability r is now played by the rate ω_m. In addition, we consider $\varphi(t) = \lambda e^{-\lambda t}$, as done in the previous section. For this specific choice, the free propagator of a mortal random walk in the Laplace space for an isotropic *velocity model* has been derived before [28] and reads as

$$P_m(x,s;x_0) = \frac{1}{2v_0}\sqrt{\frac{s+\omega_m+\lambda}{s+\omega_m}}e^{-\sqrt{(s+\omega_m)(s+\omega_m+\lambda)}|x-x_0|/v_0}, \qquad (19)$$

where the subindex m stands for *mortal*.

Using the ideas discussed above, and denoting the time convolution operator as $*$, we can write the free propagator for the random walk with resetting as

$$P(x,t;x_0,x_0^*) = P_m(x,t;x_0) + P_m(x,t;x_0^*) * \varphi(t) + P_m(x,t;x_0^*) * \varphi(t) * \varphi(t) + \dots \qquad (20)$$

or, transforming to the Laplace space,

$$P(x,s;x_0,x_0^*) = P_m(x,s;x_0) + \frac{P_m(x,s;x_0^*)\varphi(s)}{1-\varphi(s)}. \qquad (21)$$

The first term on the left-hand side of (20) accounts for the case where no resetting has occurred yet since $t = 0$, the second term stands for the case where only one resetting to the position x_0^* has occurred up to now, and so on.

Now, following the same procedure as above, we can insert Eqs. (19) and (21) into (4) to provide an expression for the first passage distribution

$$f(s;x_0,x_0^*) = \frac{s+\omega_m}{\omega_m + s\left(1+\sqrt{\frac{s+\omega_m}{s+\omega_m+\lambda}}\right)e^{\sqrt{(s+\omega_m)(s+\omega_m+\lambda)}|x_0|/v_0}} \qquad (22)$$

and for its corresponding MFPT

$$\langle T\rangle = \frac{1}{\omega_m}\left(\frac{1+\sqrt{\frac{\omega_m}{\omega_m+\lambda}}}{e^{-\sqrt{\omega_m(\omega_m+\lambda)}|x_0|/v_0}} - 1\right). \qquad (23)$$

One remarkable point is that one can recover from this result in the diffusive limit ($\lambda \to \infty$, $v_0 \to \infty$ with $v_0^2/\lambda \to D$) the expression

$$\langle T\rangle = \frac{1}{\omega_m}\left(e^{\sqrt{\omega_m/D}|x_0|} - 1\right), \tag{24}$$

which was derived for a Brownian walker with resetting in [1]. Likewise, it is easy to check from (23) that $\lim_{\omega_m\to 0}\langle T\rangle = \infty$ and $\lim_{\omega_m\to\infty}\langle T\rangle = \infty$, which guarantees the existence of an optimum resetting rate, except for the case $x_0 = 0$, where the MFPT decays monotonically with ω_m and so the minimum MFPT is obtained for $\omega_m \to \infty$. This qualitatively coincides with the result found for the case of *resetting subordinated to jumps*, which shows that in the limit $x_0 \to 0$ the optimal strategy is always resetting as much or as fast as possible. A more complete discussion on this point is provided in the next section.

Again, in the limit $\omega_m \to 0$ we obtain a scaling $\langle T\rangle \sim \omega_m^{-1/2}$, which really seems to represent a universal behavior independent of the mechanism used for resetting. This universal feature is actually reminiscent of the scaling $\sim t^{-1/2}$ for the revisit times to a given position of random walkers which have jump PDFs with finite moments. The first passage statistics is expected to be asymptotically governed by the probability to be brought back (through a reset) to the initial position after very long excursions. Specifically, MFPT should be inversely proportional to that probability, which leads to $\langle T\rangle \sim \langle t\rangle^{1/2}$, with $\langle t\rangle$ the mean time for resetting. Consequently, the scaling $\langle T\rangle \sim \omega_m^{-1/2}$ follows straightforward.

III. OPTIMAL SEARCH TIMES IN INFINITE DOMAINS

Next, we will use the results (18) and (23) just derived to analyze the existence of an optimal point in the phase space (r,λ) [or (ω_m,λ)] such that it minimizes $\langle T\rangle$ as a function of x_0. Since we have deliberately chosen a very simple scenario in which both processes (resetting and motion) are characterized by one single scale (together with the spatial scale of the problem x_0), one should expect that the optimal strategy is relatively trivial, in comparison with more complex multiscale or scale-free situations as in Lévy flights, which surprisingly lead to a first-order transition in the phase space for the optimum $\langle T\rangle$ [4]. As we will show in the following section, however, the dynamics becomes especially interesting when finite domains are considered (so a new spatial scale is introduced) with the emergence of phase transitions similar to those reported in the Lévy case.

We start by analyzing the result (18). By differentiating with respect to λ one finds that a minimum is found for

$$\lambda_{\text{opt}} = \frac{v_0}{\sqrt{r}|x_0|}\left\{1 + \mathrm{W}\left[\frac{-1}{e(1+\sqrt{r})}\right]\right\}, \tag{25}$$

where $\mathrm{W}(\ldots)$ denotes the Lambert W function. One can check that λ_{opt} is always positive for any value in the interval $0 < r \leqslant 1$. Replacing its expression into (18), the global minimum is always found for $r = 1$, which corresponds to $\lambda_{\text{opt}} = \chi v_0/|x_0|$, where we define $\chi \equiv [1 + \mathrm{W}(-0.5/e)] \approx 0.768$. So, the global strategy in this case corresponds trivially to using resetting as much as possible while adapting the flight durations to the spatial scale x_0.

Next, we go for the case of *resetting independent of motion*. Since one can easily check from (23) that the derivative of the MFPT with respect to λ is always positive, this immediately leads to the conclusion that the global optimum must be at $\lambda = 0$. Differentiation with respect to ω_m then allows one to find that the optimum satisfies $(\omega_m)_{\text{opt}} = \chi v_0/|x_0| \approx 0.768 v_0/|x_0|$. It is found then that the optimal strategy coincides with that in the case of *resetting subordinated to jumps*, and it consists of single-flight excursions interrupted by resetting at the rate $\chi v_0/|x_0|$. These results are actually very intuitive: resetting will be in general a more convenient mechanism for going back to the initial position than turning the direction of motion (at least while resetting is considered to be an instantaneous process), and consequently optimizing the MFPT should rely just on appropriately adjusting resetting to each particular situation.

We stress that additional mechanisms for resetting have been tested in which the same optimal behavior has been obtained, again with the same optimal resetting rate $\chi v_0/|x_0|$. So, we can conclude from the above that this is a universal feature for all random walk models with constant velocity v_0 provided that resetting is Markovian.

IV. OPTIMAL SEARCH TIMES IN FINITE DOMAINS

We consider now that the random walkers move in the interval $(0,L)$ with periodic boundary conditions. Note that the procedure to obtain the MFPT described in (3) and (4) is still completely valid, provided that now the free propagator $P(x,t;x_0,x_0^*)$ is replaced there by the propagator $P_L(x,t;x_0,x_0^*)$ for a finite domain of size L. Transforming from one to another is relatively simple just by explicitly imposing periodic summation techniques (see, e.g., [28,32,33])

$$P_L(x,t;x_0,x_0^*) = \sum_{m=-\infty}^{\infty} P(x+mL,t;x_0,x_0^*), \tag{26}$$

which is valid in the region of interest $x \in (0,L)$.

Introducing in (26) the expressions for the free propagators we have already derived in previous sections, one can obtain again the corresponding MFPT. Since these calculations do not provide any additional insight into the problem we will just reproduce here the final expressions obtained. For the cases of *subordinated* and *independent* resetting with $x_0^* = x_0$ the MFPT reads as, respectively,

$$\langle T\rangle = \frac{1}{\lambda r}\left[\frac{1+\sqrt{r}+(1-\sqrt{r})e^{-\lambda\sqrt{r}L/v_0}}{e^{-\lambda\sqrt{r}x_0/v_0}+e^{-\lambda\sqrt{r}(L-x_0)/v_0}} - 1\right] \tag{27}$$

and

$$\langle T\rangle = \frac{1}{\omega_m}\left[\frac{\left(1+\sqrt{\frac{\omega_m}{\omega_m+\lambda}}\right)\left(1-e^{-\sqrt{\omega_m(\omega_m+\lambda)}L/v_0}\right)}{e^{-\sqrt{\omega_m(\omega_m+\lambda)}x_0/v_0}+e^{-\sqrt{\omega_m(\omega_m+\lambda)}(L-x_0)/v_0}} - 1\right], \tag{28}$$

from which (18) and (23) can be recovered, respectively, in the limit $L \to \infty$.

The main focus of interest now is in understanding how the global optima found in the previous section for infinite domains changes as the domain size gets reduced. For this, note that in the case where the spatial scales x_0 and L are of the same order, then both resetting and changes of direction will be detrimental since coming back to x_0 is only a good strategy

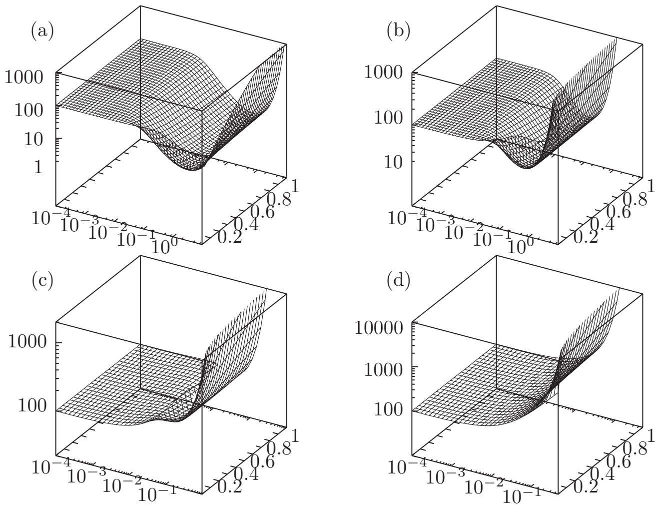

FIG. 1. MFPT for *resetting subordinated to motion* as a function of the resetting probability r and the average flight length λ for different values of the spatial scale: $x_0/L = 0.005$ (a), 0.025 (b), 0.1 (c), 0.25 (d). The legends on the right of each panel provide information about the order of magnitude of typical values of the MFPT. Arbitrary values $v_0 = 1$ and $L = 200$ have been used in all the cases.

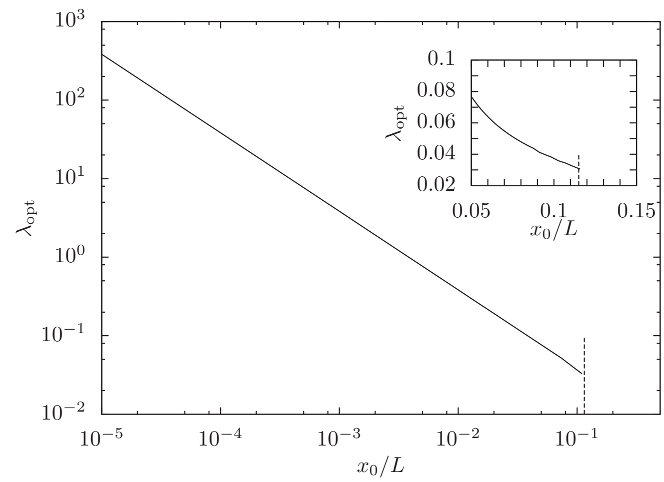

FIG. 2. Optimal value of the rate λ for the case of *resetting subordinated to motion*. The vertical dotted line represents the critical value for which the global optima becomes $(r,\lambda) = (0{,}0)$. The behavior close to the critical value is shown in the inset. Arbitrary values $v_0 = 1$ and $L = 200$ have been used in all the cases.

provided there is a target close enough to there, and this will only be the case for $x_0 \ll L$ (or, equivalently, $|L - x_0| \ll L$). Hence, as long as L decreases and that condition is not fulfilled, we should find that the optimal point in the phase space (r,λ), or (ω_m,λ), will move towards (0,0).

Interestingly, what we obtain from both (27) and (28) is that the transition from one regime to the other is not smooth but follows a first-order phase transition. Figure 1 provides plots of $\langle T \rangle$ as a function of r and λ for *resetting subordinated to motion*. There, it can be seen how the point (0,0) represents a global minimum for x_0/L large [Fig. 1(d)], but when this parameter is small an intermediate minimum appears and eventually it becomes the global minimum [see Figs. 1(a) and 1(b)]. This picture can be completed with the information provided in Figs. 2 and 3 (which correspond to the case *subordinated to motion* and *independent of motion*, respectively). There, we have computed the optimal value (of λ or ω_m, respectively) as a function of the order parameter x_0/L, and we have marked with a vertical dotted line the point at which the phase transition occurs. The scaling $\lambda_{\rm opt} \sim x_0^{-1}$ and $(\omega_m)_{\rm opt} \sim x_0^{-1}$ can be also observed there.

The critical point can be easily approximated as follows. First, one should see that the MFPT at the point (0,0) reads as $\langle T \rangle = L/2v_0$, as can be computed from (27) or (28); this coincides with the MFPT one would expect for a ballistic particle moving in the periodic domain $(0,L)$. Then, we should determine when the intermediate minimum observed in Fig. 1 takes a value equal to $L/2v_0$. For doing this, we note that the coordinates of that point can be well approximated by the optimum point we have found in the previous section (for the case of infinite domains); the justification for this is that we still expect the condition $x_0 \ll L$ to be valid close to the critical point. Then, we will use the points $(r,\lambda) = (1,\chi v_0/x_0)$ for the *subordinated* case, and $(\omega_m,\lambda) = (\chi v_0/x_0,0)$ for the *independent* case. If we replace this into (18) and (23), we obtain

$$\left(\frac{x_0}{L}\right)_{\rm cr} = \frac{\chi}{2(2e^{\chi} - 1)} \approx 0.1159 \tag{29}$$

as the critical point at which the transition occurs for both mechanisms. This result coincides well with that reported directly from Eqs. (27) and (28) in Figs. 2 and 3 (see insets there). Again, we stress that this result will be valid for any Markovian resetting mechanism provided that random walks are isotropic and the walkers move at a constant speed.

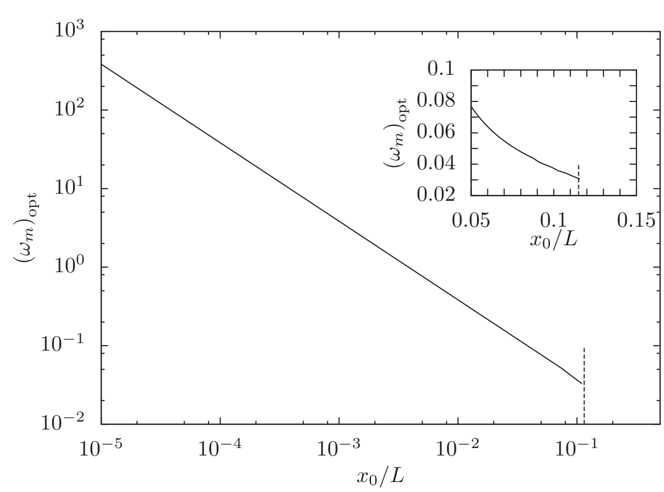

FIG. 3. Optimal value of the rate λ for the case of *resetting independent of motion*. The vertical dotted line represents the critical value for which the global optima becomes $(\omega_m,\lambda) = (0{,}0)$. The behavior close to the critical value is shown in the inset. Arbitrary values $v_0 = 1$ and $L = 200$ have been used in all the cases.

V. OPTIMAL TIMES WITH DELAYED RESETTING

We find then that our walkers with constant speed must adopt a very extreme strategy in order to optimize its search times, as they must always interrupt excursions with a reset just after the first flight is completed. One may think that this is simply because resetting is considered here to be an instantaneous process, so it is costless in terms of time. So, we could introduce a delay such that when a reset is carried out the particle must wait an average time τ before starting a new excursion. This would punish resets so a different (maybe richer) dynamics could be expected.

For the case of resetting *subordinated to motion* and *independent to motion*, we can replace the last term in Eq. (5) by

$$r\delta(x-x_0^*)\int_0^t dt'\int_{-\infty}^{\infty}dx\,\varphi(t')j(x,t-t'-\tau;x_0,x_0^*), \quad (30)$$

so we explicitly introduce the delay τ in the resetting term. With this change now the free propagator in the infinite domain [Eq. (16)] reads as

$$\begin{aligned}P(x,s;x_0,x_0^*) &= \frac{1}{2}\sqrt{\frac{s+\lambda}{s+r\lambda}}e^{-\sqrt{(s+\lambda)(s+r\lambda)}|x-x_0|/v_0}\\ &\quad+\frac{r\lambda}{2[se^{\tau s}+r\lambda(e^{\tau s}-1)]}\\ &\quad\times\sqrt{\frac{s+\lambda}{s+r\lambda}}e^{-\sqrt{(s+\lambda)(s+r\lambda)}|x-x_0^*|/v_0}. \quad (31)\end{aligned}$$

Using again Eq. (26) and repeating the same procedure as above for obtaining the MFPT, the result reached for $x_0^*=x_0$ is

$$\langle T\rangle = (1+\lambda r\tau)\langle T_0\rangle, \quad (32)$$

where $\langle T_0\rangle$ is the MFPT in the absence of the delay [given by the expression (27)]. Actually, it is possible to extend this result (not shown here) to check that (32) is also valid when the time delay is a random variable that follows any PDF with average τ.

Likewise, the case of *resetting independent of motion* leads similarly to

$$\langle T\rangle = (1+\lambda\tau)\langle T_0\rangle. \quad (33)$$

Since the delay only introduces a multiplicative factor in the computation of the MFPT, the effect of this on the optimal strategy is actually minor. One can check (Fig. 4, triangles) that the optimal point in the phase space (r,λ) for *resetting subordinated to motion* decreases when the delay τ increases, and eventually for a critical value of τ the phase transition disappears and the trivial point (0,0) remains as the unique global optima. For the *resetting independent of motion* (Fig. 4, circles), the delay does not introduce any change in the dynamics of the optimal MFPT since the parameter ω_m does not appear in the prefactor in (33). This reflects that the equivalence between both reset mechanisms studied here breaks down when additional (non-Markovian) effects as a delay effect are introduced.

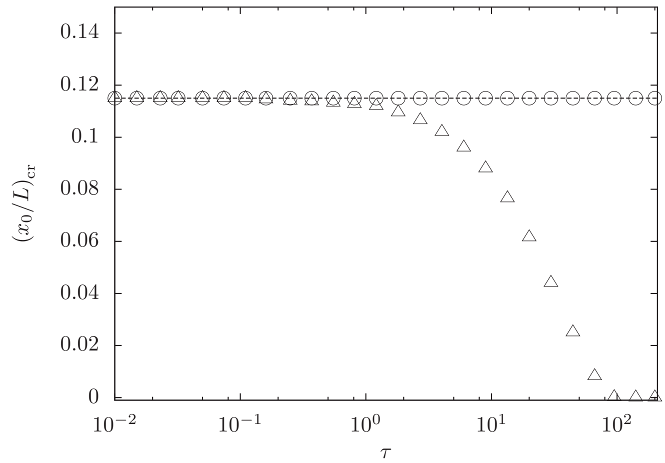

FIG. 4. Critical value of x_0/L as a function of the delay τ introduced after the reset. Results are shown both for the case of *resetting subordinated to motion* (triangles) and *resetting independent of motion* (circles) to observe the different dynamics that the delay induces in each case. Arbitrary values $v_0 = 1$ and $L = 200$ have been used in all the cases.

VI. DISCUSSION: LINKS BETWEEN RESETTING AND MULTISCALE WALKS

First passage time distributions of simple random walks in infinite media typically decay very slowly following a power-law function or similar, the most famous case being that predicted by the Sparre-Andersen theorem. Instead, resetting destroys this scaling by introducing memory effects in the process and leads eventually to an exponential decay of the first passage PDF, as typically found in finite media (albeit one must be aware that both situations are of a different nature; while the latter reaches an equilibrium state, the former asymptotically approaches a nonequilibrium stationary state). As a consequence of this, we have showed here that an optimal resetting rate (in terms of minimizing the MFPT through the origin) does appear. From that point of view, our work extends the case of diffusive movement with resetting which was already explored in Refs. [1,2] to the case of walkers moving at constant speed v_0 and carrying out exponentially distributed flights. It is so illustrative to recall the optimum resetting rate found in [1], which in our notation corresponds to $2.538v_0^2/\lambda x_0^2$. Here, we have checked that the case of exponential flights leads in all cases to an optimal rate $0.768v_0/|x_0|$. The different scaling found in the two cases comes from the fact that our random walkers follow a velocity model, so characteristic times for the motion process over a distance x are typically x/v_0 (in comparison to Brownian motion where typical times read as $x^2/D = \lambda x^2/v_0^2$).

The case of finite domains we have also explored (now with x_0 defined as positive) shows that when we introduce the additional spatial scale L, then x_0/L plays the role of an order parameter such that a first-order phase transition for the optimal resetting rate is found. So, for x_0/L small we still can approximate the optimum rate by $0.768v_0/x_0$, while above the critical point $x_0/L \approx 0.1159$ the optimum resetting rate is trivially zero. It is interesting to note that an analogous behavior has been found for the case of two-scale random walks (without resetting) in finite media [24]. If the walker is allowed to move according to two characteristic movement scales (say λ_1 and λ_2), there it was found a critical value $x_0/L \approx 0.105$ above which the global optima correspond to $\lambda_1 = \lambda_2 = 0$. Instead, for the region under the critical value, the best strategy corresponds to $\lambda_1 = 0$ and $\lambda_2 \approx 0.5v_0/x_0$. So, we observe that the role played by this second movement scale λ_2 is similar to the effect of resetting (although both mechanisms have fundamental differences since resetting clearly introduces a bias in motion towards the initial point while the two-scale walk is completely isotropic). These results reinforce the idea discussed in previous works on search theory [23,24,34] that optimizing search efficiency of random walks (in terms of minimizing their MFPT) necessarily implies an appropriate combination of different (at least two) motion scales, which in the limit case leads to the use of free-scale, i.e., Lévy, strategies. Smaller scales will then be used to efficiently *exploit* closer regions to detect closer targets, while larger scales will be used to *explore* further regions. While a more generalized and fundamental study of the link between multiscale motion and MFPT optimization is still elusive, this work provides significant evidence in this line.

ACKNOWLEDGMENTS

This research has been partially supported by Ministerio de Economia y Competitividad (MINECO) through Grant No. FIS 2012-32334 and by Generalitat de Catalunya through Grant No. SGR 2013-00923. V.M. also thanks the Isaac Newton Institute for Mathematical Sciences, Cambridge (United Kingdom) for support and hospitality during the CGP programme where part of this work was undertaken.

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