Nonstandard diffusion under Markovian resetting in bounded domains

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We consider a walker moving in a one-dimensional interval with absorbing boundaries under the effect of Markovian resettings to the initial position. The walker's motion follows a random walk characterized by a general waiting time distribution between consecutive short jumps. We investigate the existence of an optimal reset rate, which minimizes the mean exit passage time, in terms of the statistical properties of the waiting time probability. Generalizing previous results, we find that when the waiting time probability has first and second finite moments, resetting can be either (i) never beneficial, (ii) beneficial depending on the distance of the reset point to the boundary, or (iii) always beneficial. Instead, when the waiting time probability has the first or the two first moments diverging we find that resetting is always beneficial. Finally, we have also found that the optimal strategy to exit the domain depends on the reset rate. For low reset rates, walkers with exponential waiting times are found to be optimal, while for high reset rate, anomalous waiting times optimize the search process.

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I. INTRODUCTION

Brownian motion under restart has been widely studied from a theoretical point of view [1]. Depending on the type of random walk and the characteristics of the resetting mechanism, the overall process may reach an equilibrium state [2] and have an optimal strategy to reach a fixed target [3]. The mean first passage time (MFPT) of a random walker to reach a target located at a given position has been often used to determine the efficiency of resetting for different types of motion [4] and reset time distributions [5–7]. In general, the presence of a safe and fresh reset renders the walker a new opportunity to reach the target whenever it gets far from it. In the particular case where the resetting is Markovian (i.e., it restarts at a constant rate), this process has been shown to exhibit an optimal point for many types of random walk as in the case of a diffusive walker [8], subdiffusive one [9,10], performing Lévy flights [11,12] or under a combination of long-distance jumps interrupted by rests of large duration [4]. Random walks under resetting in a bounded domain have also attracted some attention. For example, in Ref. [13] the authors consider resetting for a diffusing particle moving in a bounded domain with reflecting barriers searching for a target inside the interval and in Ref [14] reactive boundaries have also been considered. Conversely, in [15,16] the exit time of a diffusing particle in a bounded domain with absorbing boundaries is studied when it resets its position to a point x_0 at a constant rate r. More specifically, they study the passage time from one of the boundaries given that the walker has not reached the other boundary. They find that the condition for which there is an optimal reset rate to exit the region of size L depends on x_0/L only, that is, it is independent of the diffusion coefficient (hence, the motion of the particle).

In this work we generalize the models proposed in Refs. [15,16] by considering a random walker whose motion follows a nonstandard diffusion under Markovian resetting. Based on the continuous-time random walk framework we consider that the walker waits a random time between successive jumps in the diffusive limit, that is, when the characteristic jump length is much lower than L. The waiting time is a random variable drawn from a given probability density. We find the criterion for an optimal reset rate which now depends not only on the combination of spatial scales x_0/L but also on the statistical properties of the waiting time density function. We provide numerical studies to support our results.

The paper is organized as follows. In Sec. II we introduce the survival probability from a continuous-time random walk perspective and we obtain the survival probability under resetting. In Sec. III the mean first exit time is studied and the conditions for which there exists an optimal reset rate are derived. Finally, we present our conclusions in Sec. IV.

II. CTRW AND SURVIVAL PROBABILITY

We consider a walker, initially located at x_0, performing a random walk in continuous time within an interval $[0, L]$ in one dimension. The walker can get absorbed by any of these boundaries. In addition, the walker is randomly reset to x_0 with a constant rate r, i.e., the resetting process is Markovian. We are interested in finding the first-passage time of the walker to see the tradeoff between the resetting and the natural absorption of the walker and how it depends on the statistical properties of the waiting time probability density function (PDF). To see this, we provide an analysis of the survival probability $S_r(x_0, t)$, defined as the probability that the walker has not hit any of the boundaries until time t, starting from any $x_0 \in [0, L]$. In other words, it estimates the probability that the walker survives (within the interval) until time t. To do this, we first need to solve the master equation for the random walk and the survival probability in the bounded domain in absence of resetting.

The random walk rules governing the walker motion are as follows. The walker starts from the initial position at $t = 0$ waiting a random time before performing the first jump to a new position. There, it waits again for a time before proceeding and so on to the next jump. Jump lengths and waiting times are independent and identically distributed random variables according to the PDFs $\Phi(x)$ and $\varphi(t)$, respectively. If $\hat{\varphi}(s)$ and $\tilde{\Phi}(k)$ are their Laplace and Fourier transforms defined by

$$\hat{\varphi}(s) = \mathcal{L}_s[\varphi(t)] = \int_0^\infty e^{-st}\varphi(t)dt$$

and

$$\tilde{\Phi}(k) = \mathcal{F}_k[\Phi(x)] = \int_{-\infty}^\infty e^{ikx}\Phi(x)dx,$$

then the PDF $P(x,t)$ for the walker position at time t is given by the Montroll-Weiss equation [17], which in the Fourier-Laplace space reads

$$\check{P}(k,s) = \frac{e^{ikx_0}[1-\hat{\varphi}(s)]}{s[1-\hat{\varphi}(s)\tilde{\Phi}(k)]}. \quad (1)$$

Note that we have used an inverted hat symbol to refer to the Fourier-Laplace transform. Rearranging Eq. (1) in the form

$$s\left[\frac{1}{\hat{\varphi}(s)} - \tilde{\Phi}(k)\right]\check{P}(k,s) = e^{ikx_0}\left[\frac{1}{\hat{\varphi}(s)} - 1\right], \quad (2)$$

after straightforward algebraic manipulations, we can rewrite the expression as

$$s\check{P}(k,s) - e^{ikx_0} = \hat{K}(s)[\tilde{\Phi}(k) - 1]\check{P}(k,s), \quad (3)$$

where we have introduced the memory kernel

$$\hat{K}(s) = \frac{s\hat{\varphi}(s)}{1-\hat{\varphi}(s)}. \quad (4)$$

If we assume that the characteristic jump distance σ is very small in comparison with the domain length L ($\sigma \ll L$), then we can consider the continuum limit of $\Phi(x)$ in the Fourier space as $\tilde{\Phi}(k) \simeq 1 - (\sigma k)^2/2$. By inverting Eq. (3) in Fourier-Laplace we obtain the master equation for nonstandard diffusion

$$\frac{\partial P(x,t)}{\partial t} = \frac{\sigma^2}{2}\int_0^t K(t-t')\frac{\partial^2 P(x,t')}{\partial x^2}dt'. \quad (5)$$

Now we are in position to solve Eq. (5) under the boundary and initial conditions

$$P(x=0,t) = P(x=L,t) = 0,$$
$$P(x,t=0) = \delta(x-x_0). \quad (6)$$

Transforming Eq. (5) by Laplace and taking into account the initial condition in Eq. (6) we obtain

$$\frac{d^2\hat{P}(x,s)}{dx^2} - \frac{2s}{\sigma^2\hat{K}(s)}\hat{P}(x,s) = -\frac{2\delta(x-x_0)}{\sigma^2\hat{K}(s)}. \quad (7)$$

This equation may be solved separately in the two regions I ($0 \leqslant x < x_0$) and II ($x_0 < x \leqslant L$). In each region separately the equation obeys the homogeneous form of Eq. (7), for which the solutions are

$$\hat{P}(x,s) = Ae^{\alpha(s)x} + Be^{-\alpha(s)x},$$

with

$$\alpha(s) \equiv \frac{1}{\sigma}\sqrt{\frac{2s}{\hat{K}(s)}} = \frac{\sqrt{2}}{\sigma}\sqrt{\frac{1}{\hat{\varphi}(s)} - 1}. \quad (8)$$

In each region we impose the corresponding boundary conditions: $P_I(x=0,t) = 0$ and $P_{II}(x=L,t) = 0$. In addition, we require that $\hat{P}(x,s)$ is continuous at $x = x_0$, i.e., $\hat{P}_I(x = x_0, s) = \hat{P}_{II}(x = x_0, s)$.

However, due to the presence of the delta function in Eq. (7) the derivative is not continuous at $x = x_0$. To find the matching condition for the derivative we integrate (7) from $x_0 - \varepsilon$ to $x_0 + \varepsilon$ (with $\varepsilon > 0$), i.e.,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{d^2\hat{P}(x,s)}{dx^2}dx - \frac{2s}{\sigma^2\hat{K}(s)}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\hat{P}(x,s)dx$$
$$= -\frac{2}{\sigma^2\hat{K}(s)}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\delta(x-x_0)dx. \quad (9)$$

Performing the integrals we have

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon}\frac{d^2\hat{P}(x,s)}{dx^2}dx$$
$$= \left(\frac{d\hat{P}(x,s)}{dx}\right)_{x=x_0+\varepsilon} - \left(\frac{d\hat{P}(x,s)}{dx}\right)_{x=x_0-\varepsilon}$$
$$= \left(\frac{d\hat{P}_{II}(x,s)}{dx}\right)_{x=x_0+\varepsilon} - \left(\frac{d\hat{P}_{I}(x,s)}{dx}\right)_{x=x_0-\varepsilon} \quad (10)$$

and

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon}\delta(x-x_0)dx = 1.$$

Taking the limit $\varepsilon \to 0$ we find

$$\lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\frac{d^2\hat{P}(x,s)}{dx^2}dx$$
$$= \left(\frac{d\hat{P}_{II}(x,s)}{dx}\right)_{x=x_0} - \left(\frac{d\hat{P}_{I}(x,s)}{dx}\right)_{x=x_0} \quad (11)$$

and

$$\lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\hat{P}(x,s)dx = 0. \quad (12)$$

Finally, plugging into Eq. (7) the matching condition reads

$$\left(\frac{d\hat{P}_{II}(x,s)}{dx}\right)_{x=x_0} - \left(\frac{d\hat{P}_{I}(x,s)}{dx}\right)_{x=x_0} = -\frac{2}{\sigma^2\hat{K}(s)}.$$

Considering the above boundary and matching conditions in the solution for each region one finds after some elementary calculations

$$\hat{P}(x,s)=\frac{1}{\sigma}\sqrt{\frac{2}{s\hat{K}(s)}}\frac{1}{\sinh[\alpha(s)L]}\times\begin{cases}\sinh[\alpha(s)(L-x_0)]\sinh(\alpha(s)x), & 0\leqslant x\leqslant x_0\\ \sinh[\alpha(s)(L-x)]\sinh(\alpha(s)x_0), & x_0\leqslant x\leqslant L\end{cases}. \tag{13}$$

The survival probability in absence of resetting $S_0(x_0,t)$ follows immediately

$$\hat{S}_0(x_0,s)=\int_0^L\hat{P}(x,s)dx=\frac{1}{s}\left\{1-\frac{\cosh\left[\alpha(s)\left(x_0-\frac{L}{2}\right)\right]}{\cosh\left[\alpha(s)\frac{L}{2}\right]}\right\}. \tag{14}$$

Finally, we can find the survival probability under resetting following Ref. [5,6] from the renewal equation

$$S_r(x_0,t)=\varphi_R^*(t)S_0(x_0,t)+\int_0^t\varphi_R(t')S_0(x_0,t')S_r(x_0,t-t')dt', \tag{15}$$

where $\varphi_R(t)$ is the PDF of reset times, that is, the probability that the time elapsed between two consecutive resets is t, and $\varphi_R^*(t)=\int_t^\infty\varphi_R(t')dt'$ is the probability that no reset has occurred before t. The first term on the right-hand side of Eq. (15) corresponds to the probability that no reset occurs before t, and the second term represents the probability that the particle resets to its initial position x_0 at least once between 0 and t. Applying the Laplace transform to Eq. (15) we obtain

$$\hat{S}_r(x_0,s)=\frac{\mathcal{L}_s[\varphi_R^*(t)S_0(x_0,t)]}{1-\mathcal{L}_s[\varphi_R(t)S_0(x_0,t)]}. \tag{16}$$

Here we will be focusing on the case where resets take place at constant rate r, so the corresponding PDF of reset times follows an exponential distribution

$$\varphi_R(t)=re^{-rt}, \tag{17}$$

and Eq. (16) can be thus rewritten as

$$\hat{S}_r(x_0,s)=\frac{\hat{S}_0(x_0,s+r)}{1-r\hat{S}_0(x_0,s+r)}, \tag{18}$$

with $\hat{S}_0(x_0,s)$ in Eq. (14).

III. MEAN FIRST EXIT TIME

The time the walker will take to reach for the first any of the boundaries in the presence of resetting at x_0 is a random variable distributed, by definition, through the PDF

$$f_r(t)=-\frac{\partial S_r(x_0,t)}{\partial t}. \tag{19}$$

The mean first exit time (MFET) is the first moment of the PDF $f_r(t)$:

$$T_r(x_0)=\langle t\rangle_{f_r}=\int_0^\infty tf_r(t)dt=\hat{S}_r(x_0,s=0)=\frac{\hat{S}_0(x_0,s=r)}{1-r\hat{S}_0(x_0,s=r)}. \tag{20}$$

A. MFET without resetting

When $r=0$ the survival probability is given by Eq. (14) and the MFET is, in analogy with Eq. (20), is as follows:

$$T_0(x_0)=\langle t\rangle_{f_0}=\hat{S}_0(x_0,s=0), \tag{21}$$

i.e., it is the first-order moment of the first exit time PDF in absence of resetting: $f_0(t)$. To compute the above limit we need to know how function $\alpha(s)$ behaves as $s\to 0$. Taking the derivative of $\alpha(s)$ with respect to s,

$$\frac{d\alpha(s)}{ds}=-\frac{1}{\alpha(s)\sigma^2\hat{\varphi}(s)^2}\frac{d\hat{\varphi}(s)}{ds}>0, \tag{22}$$

since $d\hat{\varphi}(s)/ds<0$. Then α is monotonically increasing with s, i.e.,

$$\alpha(s)\to 0\quad\text{as}\quad s\to 0. \tag{23}$$

So, for small s we can expand the factor in brackets in Eq. (14) to find

$$\hat{S}_0(x_0,s\to 0)\simeq\frac{\alpha^2(s)}{2s}x_0(L-x_0)+O[\alpha^4(s)/s]. \tag{24}$$

Likewise, if the waiting time PDF has finite moments, then its Laplace transform can be expanded for small s as

$$\frac{1}{\hat{\varphi}(s)}\simeq 1+\langle t\rangle_\varphi s+s^2\left(\langle t\rangle_\varphi^2-\frac{\langle t^2\rangle_\varphi}{2}\right)+O(s^3),$$

and from Eq. (8) one has

$$\alpha^2(s)=\frac{2}{\sigma^2}\left(\frac{1}{\hat{\varphi}(s)}-1\right)\simeq\frac{2}{\sigma^2}\langle t\rangle_\varphi s+\frac{2s^2}{\sigma^2}\left(\langle t\rangle_\varphi^2-\frac{\langle t^2\rangle_\varphi}{2}\right)+O(s^3), \tag{25}$$

where the nth order moment of the waiting time PDF is defined by

$$\langle t^n\rangle_\varphi=\int_0^\infty t^n\varphi(t)dt,\quad n\geqslant 1.$$

Plugging Eq. (25) into Eq. (24), the MFET in absence of resetting reads

$$T_0(x_0)=\frac{\langle t\rangle_\varphi}{\sigma^2}x_0(L-x_0), \tag{26}$$

if the waiting time PDF has finite moments. However, in the case of anomalous diffusion the waiting time PDF has diverging moments and the above result no longer holds. An example is the waiting time PDF

$$\varphi(t)=\frac{t^{\gamma-1}}{\tau^\gamma}E_{\gamma,\gamma}\left(-\frac{t^\gamma}{\tau^\gamma}\right), \tag{27}$$

with $0<\gamma\leqslant 1$, where

$$E_{\mu,\beta}(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\mu n+\beta)}$$

is the two parameter Mittag-Leffler function. Its Laplace transform reads

$$\hat{\varphi}(s)=\frac{1}{1+(s\tau)^\gamma}. \tag{28}$$

On setting Eq. (28) into Eq. (8), the survival probability Eq. (24) reads

$$\hat{S}_0(x_0, s \to 0) \simeq \frac{\tau x_0(L - x_0)}{\sigma^2 (s\tau)^{1-\gamma}},$$

and inverting by Laplace and taking the time derivative, the first passage PDF is

$$\begin{aligned} f_0(t) &= -\frac{\partial S_0(x_0, t)}{\partial t} \\ &\simeq \frac{\gamma x_0(L - x_0)}{\tau\sigma^2\Gamma(1-\gamma)}\left(\frac{\tau}{t}\right)^{1+\gamma} \text{ as } t \to \infty, \end{aligned} \tag{29}$$

which lacks any finite moments. Note also that from Eqs. (21) and (24) the MFET $T_0(x_0)$ diverges in this case.

B. Optimal reset rate

We explore here under which conditions there is an optimal rate r minimizing the MFET to reach any of the boundaries. From Eqs. (14) and (20) the MFET in presence of resetting is

$$T_r(x_0) = \frac{1}{r}\left[\frac{\cosh\left(\frac{\alpha(r)L}{2}\right)}{\cosh\left(\alpha(r)\left(x_0 - \frac{L}{2}\right)\right)} - 1\right]. \tag{30}$$

When the reset rate is very large the walker does not have enough time to diffuse and reach the boundaries, then it is expected that the MFET increases as $r \to \infty$. This can be mathematically checked by virtue of Eq. (22), from which one observes that $\alpha(r) \to +\infty$ as $r \to \infty$. Then, approximating the hyperbolic functions for large argument we find that the MFET behaves as

$$T_r(x_0) \sim \frac{e^{\alpha(r)(L-x_0)}}{r} \quad \text{when} \quad r \to \infty. \tag{31}$$

This shows that the MFET increases unboundedly as r does, regardless the form of the waiting time PDF. Hence, Eq. (30) will necessarily reach a minimum for a specific rate r if the condition

$$\left(\frac{dT_r(x_0)}{dr}\right)_{r=0} < 0 \tag{32}$$

is fulfilled. In summary, the MFET will reach a minimum if it is an increasing function of r for large r and decreasing function of r for small r. The first one of these conditions is always satisfied, but the second one depends on the statistical properties of the waiting time PDF. Equation (32), then, is equivalent to the condition $\mathsf{C_V}(f_0) > 1$ for the coefficient of variation of the first exit time PDF in absence of resetting defined as [18]

$$\mathsf{C_V}(f_0) = \frac{\sqrt{\langle t^2\rangle_{f_0} - \langle t\rangle_{f_0}^2}}{\langle t\rangle_{f_0}}. \tag{33}$$

This condition is universal in the sense that is independent of the underlying random walk. The value of r which minimizes $T_r(x_0)$ is the optimal reset rate and it can be regarded as the order parameter of a Landau-like theory of phase transition in resetting systems [19]. In addition, this criterion is actually a ramification of a very generic idea, namely, the inspection paradox which ubiquitously appears in many different areas of physics [20]. To understand how this condition is satisfied in terms of the waiting time PDF we need to analyze separately the cases of waiting time PDFs with both first and second moments finite, only with first moment finite or with both moments diverging.

1. *Waiting time PDF with finite first and second moments*

Taking the limit $r \to 0$ in Eq. (30) and using Eqs. (25) and (26) we find

$$\frac{T_r(x_0)}{T_0(x_0)} \simeq 1 + r\langle t\rangle_\varphi\left(1 - \frac{\langle t^2\rangle_\varphi}{2\langle t\rangle_\varphi^2} - \frac{L^2 - 5x_0L + 5x_0^2}{6\sigma^2}\right) + O(r^2),$$

where the expansion of $T_r(x_0)$ has been normalized dividing by $T_0(x_0)$. Note that this does not affect condition Eq. (32), which reads

$$\frac{1 - 5x_0/L + 5x_0^2/L^2}{6(\sigma/L)^2} > 1 - \frac{\langle t^2\rangle_\varphi}{2\langle t\rangle_\varphi^2}, \tag{34}$$

holding for any waiting time PDF with finite both first and second moments. This condition is a criterion for the existence of an optimal reset rate. It could be also obtained from Eq. (33) using the survival probability given in Eq. (14).

When the waiting time PDF is exponential, i.e., the walker's motion follows a standard diffusion, the right-hand side of Eq. (34) is identically zero and we recover the condition derived in Ref. [15] [see Eq. (17)]. Unlike the case of standard diffusion where the optimal reset rate condition Eq. (34) depends on the spatial scales x_0 and L only, in general it depends on the first and second moments of the waiting time PDF. The term of the right-hand side of the inequality Eq. (34) may be positive, negative or zero. To discuss the possible situations in terms of the statistical properties of the waiting time PDF let us introduce the coefficient of variation $\mathsf{C_V}(\varphi)$ or relative standard deviation

$$\mathsf{C_V}(\varphi) = \frac{\sqrt{\langle t^2\rangle_\varphi - \langle t\rangle_\varphi^2}}{\langle t\rangle_\varphi}.$$

Then, from Eq. (34) we find the following cases:

(i) If $\mathsf{C_V}(\varphi)^2 > 1 + \frac{1}{12}(L/\sigma)^2$ condition Eq. (34) is satisfied for any value of x_0/L, then an optimal reset rate always exists. In this case, resetting is always beneficial.

(ii) If $1 - \frac{1}{3}(L/\sigma)^2 < \mathsf{C_V}(\varphi)^2 < 1 + \frac{1}{12}(L/\sigma)^2$, then the right-hand side of condition Eq. (34) is positive, and an optimal reset rate exists if and only if

$$\frac{x_0}{L} \notin \left(\frac{5 - \sqrt{5 + \Delta}}{10}, \frac{5 + \sqrt{5 + \Delta}}{10}\right), \tag{35}$$

with

$$\Delta = 120\left(\frac{\sigma}{L}\right)^2\left(1 - \frac{\langle t^2\rangle}{2\langle t\rangle^2}\right).$$

In this case, resetting is beneficial depending on the values of x_0/L.

(iii) If $\mathsf{C_V}(\varphi)^2 < 1 - \frac{1}{3}(L/\sigma)^2$ condition Eq. (34) is never satisfied, then no optimal reset rate exists, regardless of the values of x_0/L. In this case, resetting is never beneficial.

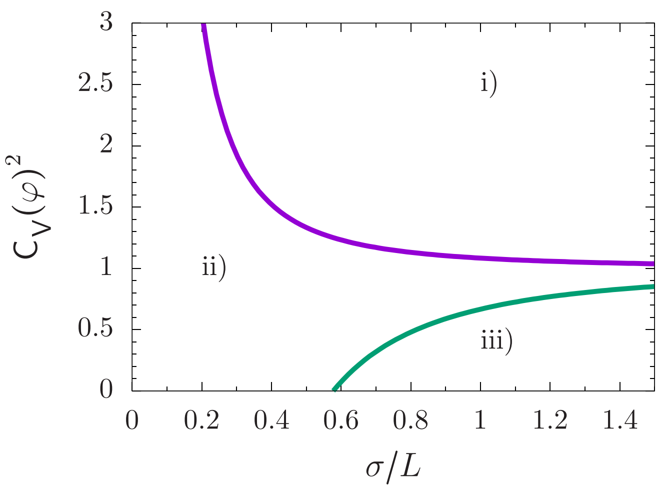

FIG. 1. Parameter space diagram for the different existence regions of optimal reset rate.

However, this situation is hardly satisfied since we have assumed [see the text below Eq. (4)] that the characteristic jump distance is much lower than the domain length, i.e., $\sigma/L \ll 1$. In Fig. 1 we plot the different cases in a parameter space.

To explore the above results we choose a specific waiting time PDF which depends on a parameter m. Let us consider the PDF

$$\varphi(t) = \frac{(t/\tau)^m}{\tau\Gamma(m+1)} e^{-t/\tau}, \quad m \geqslant 0, \tag{36}$$

which has all moments finite. This includes the exponential distribution by taking $m = 0$. The coefficient of variation reads in this case

$$\mathsf{C_V}(\varphi)^2 = \frac{1}{m+1}.$$

For this waiting time PDF the existence condition for optimal reset rate Eq. (34) corresponds to the case (ii), regardless of m.

Another interesting case is the Pareto PDF

$$\varphi(t) = \frac{\alpha}{\tau\left(1+\frac{t}{\tau}\right)^{1+\alpha}}, \quad \alpha > 0. \tag{37}$$

It has finite moments up to order n if $\alpha > n$, then Eq. (34) holds if $\alpha > 2$. The coefficient of variation is

$$\mathsf{C_V}(\varphi)^2 = \frac{\alpha}{\alpha - 2},$$

so that an optimal reset rate always exists regardless of the values of x_0/L [case (i)] if $2 < \alpha < 2 + 24(\sigma/L)^2$, which corresponds to a very narrow case where α is higher than 2 but very close to 2 provided that $\sigma \ll L$. The optimal reset rate exists depending on the values of x_0/L [case (ii)] if $\alpha > 2 + 24(\sigma/L)^2$.

2. *Waiting time PDF with diverging moments*

When the first, second, or both moments of the waiting time PDF do not exist then condition Eq. (34) does not hold in determining the existence of an optimal reset rate. This is, for example, the relevant case when the walker's motion is subdiffusive. To explore the behavior of $T_r(x_0)$ in the small r limit we need to choose specify expressions of $\varphi(t)$. Once the existence of a minimum MFET is shown we can find an exact approximated expression for the optimal reset rate as we show in turn. If we expand the denominator in Eq. (30) by using the properties of the hyperbolic trigonometric functions, then Eq. (30) adopts the form

$$T_r(x_0) = \frac{1}{r}\left\{\frac{1}{\cosh[\alpha(r)x_0] - \sinh[\alpha(r)x_0]\tanh[\alpha(r)\frac{L}{2}]} - 1\right\}.$$

Now we approximate this expression by considering that in the continuum spatial limit the characteristic jump size σ is very small in comparison with domain size L, i.e., $L \gg \sigma$. Since $\alpha(r)$ goes as σ^{-1} [see Eq. (8)] the argument of the hyperbolic tangent is very large so that $\tanh[\alpha(r)L/2] \simeq 1$. Finally, taking into account that $\cosh[\alpha(r)x_0] - \sinh[\alpha(r)x_0] = e^{-\alpha(r)x_0}$ the MFET reads

$$T_r(x_0) \simeq \frac{1}{r}(e^{\alpha(r)x_0} - 1) \quad \text{as} \quad L \gg \sigma. \tag{38}$$

Taking the derivative with respect to r equal to zero we find the equation whose solution is the optimal reset rate r_{opt}:

$$\left[r\frac{d\alpha(r)}{dr}\right]_{r=r_{\text{opt}}} \simeq \frac{1}{x_0}. \tag{39}$$

Unfortunately, this equation cannot be solved in general and we need to consider specific expressions for the waiting time PDF. Some of them are discussed in the following sections.

3. *Waiting time PDF with finite first moment and diverging second moment*

Let us consider the Pareto PDF in Eq. (37). It has first moment finite and second moment diverging for $1 < \alpha < 2$. To prove the existence of the optimal reset rate we analyze the MFET in the limit $r\tau \to 0$. The Laplace transform of the Pareto PDF can be expressed as

$$\hat{\varphi}(r) = \alpha U(1, 1-\alpha; r\tau),$$

where $U(a, b; z)$ is the Tricomi confluent hypergeometric functions of the second kind [21]. Expanding this function for small argument (using Eqs. (13.1.3) and (13.1.2) of Ref. [21]) one finds

$$U(1, 1-\alpha; r\tau) \simeq \frac{1}{\alpha} - \frac{r\tau}{\alpha(\alpha-1)} - \frac{\pi(r\tau)^\alpha}{\sin(\pi\alpha)\Gamma(1+\alpha)} + O[(r\tau)^2],$$

which inserted into the expression for $\alpha(r)$ in Eq. (8) yields

$$\alpha(r) \simeq \frac{\sqrt{2}}{\sigma}\sqrt{\frac{r\tau}{\alpha-1}}\left[1 + \frac{\pi(\alpha-1)}{2\sin(\pi\alpha)\Gamma(1+\alpha)}(r\tau)^{\alpha-1} + \dots\right]. \tag{40}$$

However, since $\alpha(r) \sim (r\tau)^{1/2}$ in the limit $r\tau \to 0$, then $\alpha(r)$ is also very small in this limit. Approximating the hyperbolic functions in Eq. (30) for small $\alpha(r)$ we find

$$T_r(x_0) \simeq \frac{x_0(L-x_0)}{2r}\alpha(r)^2, \tag{41}$$

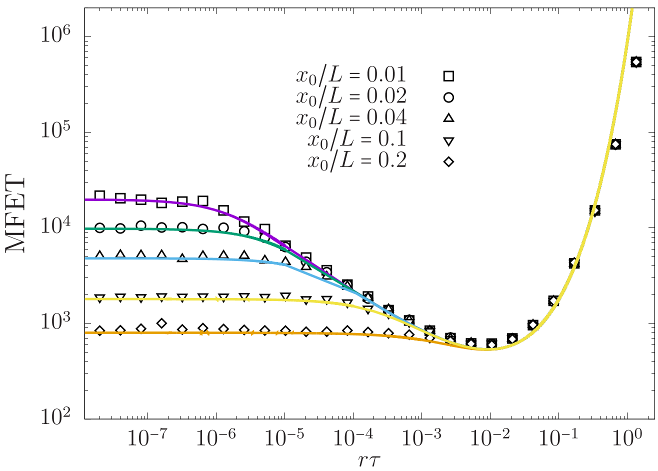

FIG. 2. Plot for the MFET vs $r\tau$ for different values of x_0/L when the waiting time PDF is a Pareto PDF with $\alpha = 3/2$. Solid curves correspond to theoretical result in Eq. (30) while symbols are the numerical simulations. $L = 1$ and $\sigma = 0.1x_0$. When $r \to 0$ the MFET tends to a finite value given by taking this limit to Eq. (41), that is $T_0(x_0) = x_0(L - x_0)\tau/\sigma^2(\alpha - 1)$.

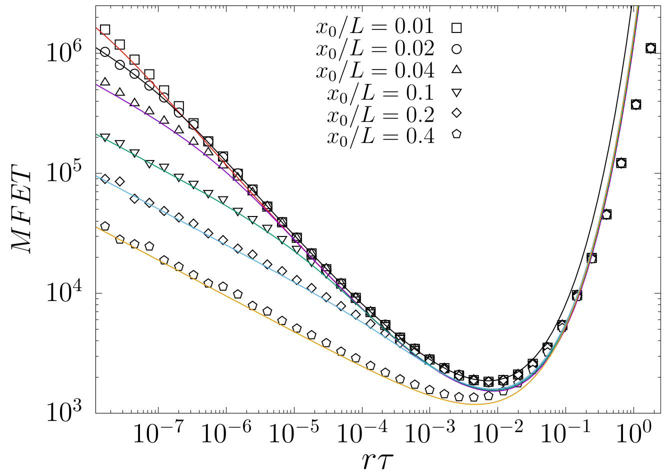

FIG. 3. Plot for the MFET vs $r\tau$ for different values of x_0/L when the waiting time PDF is a Mittag-Leffler function with $\gamma = 0.7$. Solid curves correspond to theoretical result in Eq. (43) while symbols are the numerical simulations. $L = 1$ and $\sigma = 0.1x_0$. In the limit $r \to 0$ from Eq. (44) $T_r(x_0) \sim r^{-1+\gamma}$, i.e., the MFET diverges, although it cannot be observed in the figure because the lowest value shown is 10^{-8}.

and inserting Eq. (40) one readily finds the MFET for small reset rate

$$T_r(x_0) \simeq \frac{x_0(L - x_0)}{\sigma^2}\tau\left[\frac{1}{\alpha - 1} + \frac{\pi(r\tau)^{\alpha-1}}{\sin(\pi\alpha)\Gamma(1+\alpha)}\right].$$

Finally, taking the derivative with respect to r we obtain

$$\frac{\partial T_r(x_0)}{\partial r} \simeq \frac{x_0(L - x_0)\tau^2}{\sigma^2}\frac{\pi(\alpha - 1)}{(r\tau)^{2-\alpha}\sin(\pi\alpha)\Gamma(1+\alpha)} < 0,$$

since $\sin(\pi\alpha) < 0$ for $1 < \alpha < 2$. In consequence, Eq. (32) is fulfilled and there is an optimal reset rate, i.e., resetting will always be beneficial when the waiting time PDF has finite first moment but diverging second moment. The optimal reset rate could be found from Eq. (39) but it involves Tricomi functions and it should be numerically solved. In Fig 2 we compare the numerical simulations of the MFET (in symbols) with the exact analytical result (solid lines) obtained by introducing

$$\alpha(r) = \frac{\sqrt{2}}{\sigma}\sqrt{\frac{1}{\alpha U(1, 1-\alpha; r\tau)} - 1}$$

into Eq. (30). It is shown that an optimal reset time exists for different values of x_0/L. As can be appreciated the agreement is excellent.

4. *Waiting time PDF with diverging first and second moments*

We explore now the existence of the optimal resetting rate when the two first moments of waiting time PDF are diverging. This is the case of the Pareto PDF for $0 < \alpha < 1$ or for the Mittag-Leffler PDF (27). We consider here the latter because it allows us to recover the exponential PDF case for $\gamma = 1$. Inserting Eq. (28) into Eq. (8) we find

$$\alpha(r) = \frac{\sqrt{2}}{\sigma}(r\tau)^{\gamma/2} \tag{42}$$

and from Eq. (30) we obtain

$$T_r(x_0, \gamma) = \frac{1}{r}\left\{\frac{\cosh\left[\frac{\sqrt{2}L}{2\sigma}(r\tau)^{\frac{\gamma}{2}}\right]}{\cosh\left[\frac{\sqrt{2}(x_0 - L/2)}{\sigma}(r\tau)^{\frac{\gamma}{2}}\right]} - 1\right\}. \tag{43}$$

To analyze the behavior of the MFET near $r = 0$ we note that $\alpha(r) \sim (r\tau)^{\gamma/2}$ so that $\alpha(r)$ tends to zero when r does. Then Eq. (41) holds, i.e.,

$$T_r(x_0, \gamma) \simeq \frac{\tau}{(r\tau)^{1-\gamma}}\frac{x_0(L - x_0)}{\sigma^2}, \tag{44}$$

which diverges at $r = 0$. In consequence, if the waiting time PDF has both first and second moments diverging, then resetting will always be beneficial because there always exists an optimal reset rate.

In Fig. 3 we plot the MFET showing that there is again an optimal reset rate for all the values of x_0/L, something that have verified by numerical simulations. Unlike the previous case, the optimal reset rate can be exactly computed by plugging Eq. (42) into Eq. (39) to get

$$r_{\mathrm{opt}} \simeq \frac{1}{\tau}\left(\frac{\sqrt{2}\sigma}{\gamma x_0}\right)^{2/\gamma}. \tag{45}$$

In Fig. 4 we compare Eq. (45) with the numerical simulations; note that due to the spatial symmetry in the domain, taking a value for x_0 close to 0, namely, $x_0 = \epsilon \ll 1$, is equivalent to $x_0 = L - \epsilon$. In general, the agreement observed in Fig. 4 is excellent, but for small values of γ the waiting times are eventually very large and this requires very high computational time which reduces the number of realizations and the accuracy. It is remarkable, in particular, the nonmonotonic behavior exhibited by r_{opt}, and the fact that for γ small the optimal reset rate can get modified by several orders of magnitude just by slightly changing the reset position x_0. This is a nontrivial consequence of the interplay between the

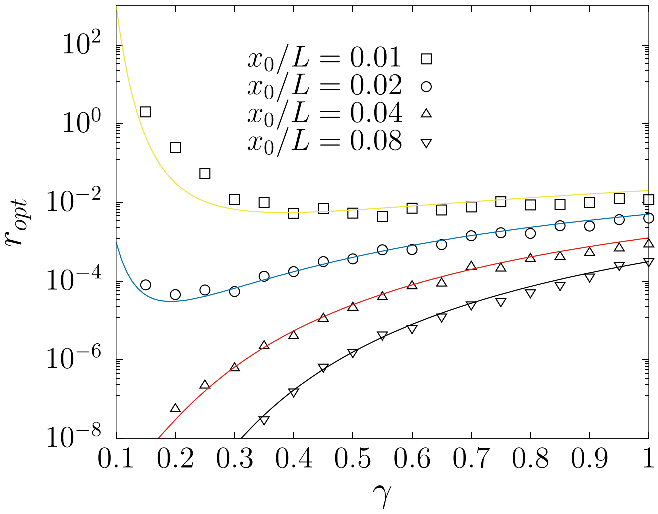

FIG. 4. Plot for the optimal reset rate $r_{\rm opt}$ vs. the anomalous exponent γ (for $\sigma = 10^{-3}$ and $L = 1$). The solid curves correspond to the analytical result given by Eq. (45).

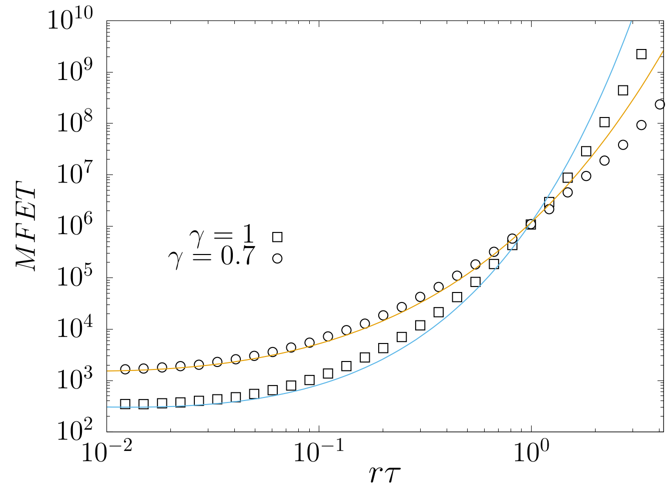

FIG. 5. Plot for $T_r(x_0, \gamma)$ vs $r\tau$. Symbols correspond to numerical simulations and solid curves are the theoretical predictions given by Eq. (43). $L = 1$, $x_0 = 0.1$, and $\sigma = 0.01$.

diverging waiting times (which lead to a diverging MFET, as seen in Sec. III A) and the timescale for resetting, r^{-1}. There is a necessity to include resettings to stop long waiting times, but this must be done without compromising too much the probability to reach the boundary.

Finally, note that inserting Eq. (45) into Eq. (43) the minimal MFET reads

$$T_r^* \simeq \tau\left(\frac{\gamma x_0}{\sqrt{2}\sigma}\right)^{2/\gamma} e^{2/\gamma} \sim \left(\frac{x_0}{\sigma}\right)^{2/\gamma}. \quad (46)$$

C. Optimal search strategy

Next we investigate which is the search strategy which has a lower MFET for a given reset rate r, that is, what is the waiting time PDF (exponential or Mittag-Leffler) that has a lower MFET for a given r. To do this we compare $T_r(x_0, \gamma)$ with $T_r(x_0, \gamma = 1)$. For fixed x_0, L, and γ, the equation $T_r(x_0, \gamma) = T_r(x_0, \gamma = 1)$ has a unique solution for $r\tau = 1$. Defining $r\tau = 1 - \epsilon$ with $|\epsilon| \ll 1$, the quotient between both MFETs can be expanded about $r\tau = 1$

$$\frac{T_r(x_0, \gamma)}{T_r(x_0, \gamma = 1)} \simeq 1 + \frac{1-\gamma}{\sigma\sqrt{2}} A\epsilon + O(\epsilon^2),$$

where

$$A = a\frac{(L - x_0)b(a^2 - 1) + x_0(b^2 - a^2)}{(a - 1)(b - a)(a^2 + b)}$$

and

$$a = e^{\sqrt{2}x_0/\sigma}, \ \ b = e^{\sqrt{2}L/\sigma}.$$

Clearly, $1 < a < b$, so that $A > 0$. From this we conclude that if $r\tau < 1$ (low reset rates) (i.e., $\epsilon > 0$), then

$$T_r(x_0, \gamma) > T_r(x_0, \gamma = 1),$$

and the search strategy with exponential waiting time is optimal. If $r\tau > 1$ (high reset rates) (i.e., $\epsilon < 0$), then

$$T_r(x_0, \gamma) < T_r(x_0, \gamma = 1),$$

and the search strategy with anomalous waiting time is the optimal. This is confirmed by numerical simulations as shown in Fig. 5. While the crossing point is robustly found at $r\tau = 1$, note that simulations for $r\tau > 1$ are not accurate because in this regime the reset process is too fast compared to the characteristic time that the walker needs to reach the boundary from x_0. In consequence, the walker spends all of the time resetting again and again until by chance an extreme value of the resetting time appears and so it opens the possibility for the walker to reach the boundary. As a result of this dynamics, the computing time for simulations increase exponentially with r, so that the number of realizations that can be computed in a reasonable time reduces very much, so compromising the accuracy of the results.

IV. CONCLUSIONS

By generalizing the model studied in Ref. [15] for a diffusive random walk in a finite interval with resets, we have been able to explore interesting, and previously unreported, regimes of optimal behavior of the MFET as a function of the reset rate. A formal expression for the mean exit time has been obtained for general waiting time [Eq. (30)] and jump length distributions in the limit where the length of the interval L is large in comparison with the jump distance σ. From that, we find that the existence of an optimal reset rate to exit the interval depends exclusively on the spatial scales only for a very particular choice of the waiting time distribution of the walker (exponential distribution). Conversely, whenever the waiting times are not Markovian, the optimality of resetting depends also on the statistical properties of the waiting time distribution of the walker. Finally, we have also found that the optimal strategy to exit the interval given a reset rate r depends on the rate itself. For low reset rates, walkers with exponential waiting times are found to be optimal and, when resetting is more frequent, anomalous waiting times optimize the process. These results open new questions as whether the casuistic herein found would hold for a more general non-Markovian resetting. Also, a general investigation of a mixed absorbing-reflecting boundaries is still lacking in the resetting literature.

[1] M. R. Evans, S. N. Majumdar, and G. Schehr, J. Phys. A: Math. Theor. **53**, 193001 (2020).
[2] V. Méndez and D. Campos, Phys. Rev. E **93**, 022106 (2016).
[3] D. Campos and V. Méndez, Phys. Rev. E **92**, 062115 (2015).
[4] V. Méndez, A. Masó-Puigdellosas, T. Sandev, and D. Campos, Phys. Rev. E **103**, 022103 (2021).
[5] A. Chechkin and I. M. Sokolov, Phys. Rev. Lett. **121**, 050601 (2018).
[6] A. Masó-Puigdellosas, D. Campos, and V. Méndez, Phys. Rev. E **99**, 012141 (2019).
[7] A. S. Bodrova and I. M. Sokolov, Phys. Rev. E **101**, 062117 (2020).
[8] M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. **106**, 160601 (2011).
[9] L. Kuśmierz and E. Gudowska-Nowak, Phys. Rev. E **99**, 052116 (2019).
[10] J. Masoliver and M. Montero, Phys. Rev. E **100**, 042103 (2019).
[11] L. Kuśmierz and E. Gudowska-Nowak, Phys. Rev. E **92**, 052127 (2015).
[12] S. N. Majumdar, P. Mounaix, S. Sabhapandit, and G. Schehr, J. Phys. A: Math. Theor. **55**, 034002 (2022).
[13] C. Christou and A. Schadschneider, J. Phys. A: Math. Theor. **48**, 285003 (2015).
[14] A. Pal, I. P. Castillo, and A. Kundu, Phys. Rev. E **100**, 042128 (2019).
[15] A. Pal and V. V. Prasad, Phys. Rev. E **99**, 032123(2019).
[16] X. Durang, S. Lee, L. Lizana, and J.-H. Jeon, J. Phys. A: Math. Theor. **52**, 224001 (2019).
[17] E. W. Montroll and G. H. Weiss, J. Math. Phys. **6**, 167 (1965).
[18] A. Pal and S. Reuveni, Phys. Rev. Lett. **118**, 030603 (2017).
[19] A. Pal and V. V. Prasad, Phys. Rev. Res. **1**, 032001(R) (2019).
[20] A. Pal, S. Kostinski, and S. Reuveni, J. Phys. A: Math. Theor. **55**, 021001 (2022).
[21] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs and Mathematical Tables* (Dover, Mineola, NY, 1964).