First-passage times for generalized heterogeneous telegrapher's processes

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We consider two different fractional generalizations of the heterogeneous telegrapher's process with and without stochastic resetting. Both governing fractional heterogeneous telegrapher's equations can be obtained from the corresponding standard heterogeneous telegrapher's equations by using the subordination approach. The first-passage time problems are solved analytically for both models by finding the survival probabilities, the first-passage time densities, and the mean first-passage times. We showed that for both cases there are optimal resetting rates for which the mean first-passage times are minimal. The present work carries implications toward our understanding of anomalous diffusion and random search in heterogeneous media.

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I. INTRODUCTION

Diffusion processes in heterogeneous environments often become anomalous, either due to the crowded environment in which the particle is immersed [1–3], due to the fractal structure [4] or the geometric constraints of the environment [5], due to the variation of the local diffusion coefficient in time [6] and space [7], or due to the external random forces that act on the particle [8,9]. This means that the mean squared displacement (MSD) of the particle has a power-law dependence on time, $\langle x^2(t)\rangle \sim t^\beta$. Depending on the anomalous diffusion exponent β the corresponding process may be subdiffusion for $0<\beta<1$, normal diffusion for $\beta=1$, and superdiffusion for $\beta>1$ [10]. There are many different approaches to anomalous diffusion, such as continuous time random walk models [11], fractional diffusion and Fokker-Planck equations [12,13], generalized (fractional) Langevin equations [14,15], heterogeneous diffusion and Langevin equations [16–19], time-dependent diffusivities [20], as well as position and time-dependent drift [21], to name but a few. Such equations are derived starting from some random process, for example, either by introducing long-tailed waiting times and jump lengths in the continuous time random walk model [10], by introducing power-law correlations in time in the driving force [14,15,22], or by using multiplicative random driving noise [17], etc. All of these processes are generalizations of the standard Brownian motion which can be described by the standard diffusion equation for the probability density function (PDF) and by the stochastic Langevin equation for the particle trajectory, and for which the MSD has a linear dependence on time.

Another class of processes is the so-called telegrapher's processes [23,24] which are generalization of the standard diffusion process in which the velocity of the particle is finite. In such processes in the short time limit the particle performs ballistic motion ($\langle x^2(t)\rangle \sim t^2$) before it turns to normal diffusion in the long time limit. The telegrapher's process can be modeled by using the telegrapher's equation in which additionally to the diffusion equation a second derivative of the PDF in time occurs [25]

$$\tau\frac{\partial^2}{\partial t^2}P_0(x,t)+\frac{\partial}{\partial t}P_0(x,t)=D\frac{\partial^2}{\partial x^2}P_0(x,t), \tag{1}$$

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where τ is a time parameter, D is the diffusion coefficient, and $v = \sqrt{D/\tau}$ is the particle velocity. This telegrapher's process can also be modeled by the Langevin equation [26]

$$\dot{x}(t) = v\zeta(t), \tag{2}$$

where $v > 0$ is the speed of the particle, while $\zeta(t)$ is a stationary dichotomic Markov process that switches between two values ± 1 with a mean rate ν, such that $\tau = \frac{1}{2\nu}$. In the limit $\tau \to 0$ and $v \to \infty$ such that $v^2\tau$ fixed, the dichotomic noise reduces to white noise.

In this paper, we will generalize the standard telegrapher's process to a process in a heterogeneous environment. One way to derive the heterogeneous telegrapher's process is by using the Langevin equation with multiplicative dichotomic noise

$$\dot{x}(t) = v(x)\zeta(t), \tag{3}$$

where $v(x) > 0$ is a position-dependent speed, $v(x) = \sqrt{D(x)/\tau}$ [27], $D(x)$ is a position-dependent diffusion coefficient, and $\zeta(t)$ is again a stationary dichotomic Markov process, as in the standard telegrapher's process. The corresponding equation for the probability density function $P_1(x,t)$ reads [27] (see also [28–30])

$$\tau\frac{\partial^2}{\partial t^2}P_1(x,t) + \frac{\partial}{\partial t}P_1(x,t) = \frac{\partial}{\partial x}\left\{\sqrt{D(x)}\frac{\partial}{\partial x}[\sqrt{D(x)}P_1(x,t)]\right\}. \tag{4}$$

Another way to define the heterogeneous telegrapher's process can be by using the generalized master equation for the continuous time random walk [31]:

$$\frac{\partial}{\partial t}P_2(x,t) = \frac{\partial}{\partial x}\int_0^t K(t-t')D(x)\frac{\partial}{\partial x}P_2(x,t')\,dt', \tag{5}$$

when the memory kernel $K(t)$ is exponential, i.e., $K(t) = \frac{1}{\tau}e^{-t/\tau}$. Thus, one obtains the following heterogeneous telegrapher's equation (HTE):

$$\tau\frac{\partial^2}{\partial t^2}P_2(x,t) + \frac{\partial}{\partial t}P_2(x,t) = \frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}P_2(x,t)\right]. \tag{6}$$

Such an equation can also be derived for a current in a lossy transmission inhomogeneous line [32].

Equations (4) and (6) can be cast in the general form

$$\tau\frac{\partial^2}{\partial t^2}P(x,t) + \frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left\{D(x)^{1-A/2}\frac{\partial}{\partial x}[D(x)^{A/2}P(x,t)]\right\}, \tag{7}$$

which for $A = 1$ corresponds to the HTE (4), while for $A = 0$ corresponds to the HTE (6). For $\tau = 0$ this equation reduces to the heterogeneous diffusion equation, which was considered in [16,17,19,33–35] and depending on the noise interpretation there are three different forms of the heterogeneous diffusion equation, such as Stratonovich for $A = 1$, Hänggi-Klimontovich for $A = 0$, and Itô form for $A = 2$. The diffusion coefficient used in this work is of power law form $D(x) = D_\alpha|x|^\alpha$, $\alpha < 2$ [17]. For $\alpha = 0$, it is the standard telegrapher's equation.

Here, we note that another form of the heterogeneous telegrapher's equation can be derived from the persistent random walk in an inhomogeneous medium [36], which has the form of Eq. (4) but $\sqrt{D(x)}$ would occur in front of the first partial derivative with respect to x instead of multiplying the PDF. Moreover, there are different generalizations of the telegrapher's process, such as the fractional telegrapher's processes [37–44], generalized discrete-time telegrapher's process [45,46], as well as the corresponding processes with random velocities [47,48]. In this paper, we will consider a fractional generalization of heterogeneous telegrapher's processes with position-dependent diffusion coefficient, governed by Eqs. (4) and (6). Such generalizations could be of importance for a description of the run-and-tumble motion of bacteria [49–54] and transient super-diffusion displayed by the multipotent progenitor cells [55] and in an analysis of generalization of different run-and-tumble motions [56–61].

This paper is organized as follows. In Sec. II, we derive the fractional heterogeneous telegrapher's equations (FHTE) from the standard equations by using the subordination approach. The first-passage properties of both models are analyzed in Sec. III. In Sec. IV, we consider both processes in the presence of stochastic resetting. We find analytical results for the mean first-passage time and show that there is an optimal resetting rate in both models at which the mean first-passage time is minimal. A summary is provided in Sec. V. In the Appendixes, we provide detailed calculations for solving the FHTEs, definitions, and some useful properties and relations of the Fox H-function and the Mittag-Leffler functions.

II. DERIVATION OF THE FHTE

We use the subordination integral [13,62,63]

$$P_s(x,t) = \int_0^\infty P(x,u)h(u,t)\,du, \tag{8}$$

where $h(u,t)$ is the subordination function for the μ-stable Lévy subordinator, which in Laplace space reads

$$\hat{h}(u,s) = \tau^{\mu-1}s^{\mu-1}e^{-u\tau^{\mu-1}s^{\mu}}, \quad 0 < \mu < 1, \tag{9}$$

to derive the corresponding equation for the fractional heterogeneous telegrapher's process (FHTP). We can show that the subordinated PDF is a solution to the equation (see Appendix A)

$$\tau^{\mu}\frac{\partial^{2\mu}}{\partial t^{2\mu}}P_s(x,t) + \frac{\partial^{\mu}}{\partial t^{\mu}}P_s(x,t) = \frac{\partial}{\partial x}\left\{\mathcal{D}(x)^{1-A/2}\frac{\partial}{\partial x}[\mathcal{D}(x)^{A/2}P_s(x,t)]\right\}. \tag{10}$$

Therefore, for $A = 1$, we obtain the following time fractional heterogeneous telegrapher's equation [30]:

$$\tau^{\mu}\frac{\partial^{2\mu}}{\partial t^{2\mu}}P_{s,1}(x,t) + \frac{\partial^{\mu}}{\partial t^{\mu}}P_{s,1}(x,t) = \frac{\partial}{\partial x}\left\{\sqrt{\mathcal{D}(x)}\frac{\partial}{\partial x}[\sqrt{\mathcal{D}(x)}P_{s,1}(x,t)]\right\}, \tag{11}$$

while for $A = 0$, it has the form

$$\tau^{\mu}\frac{\partial^{2\mu}}{\partial t^{2\mu}}P_{s,2}(x,t)+\frac{\partial^{\mu}}{\partial t^{\mu}}P_{s,2}(x,t)=\frac{\partial}{\partial x}\left[\mathcal{D}(x)\frac{\partial}{\partial x}P_{s,2}(x,t)\right], \tag{12}$$

where $\mathcal{D}(x)=\mathcal{D}_{\alpha}|x|^{\alpha}$, $\mathcal{D}_{\alpha}=\tau^{1-\mu}D_{\alpha}$, and $\frac{\partial^{\nu}}{\partial t^{\nu}}$ is the Caputo fractional derivative of order ν, defined by [64]

$$\frac{\partial^{\nu}}{\partial t^{\nu}}f(t)=\frac{1}{\Gamma(n-\nu)}\int_0^t(t-t')^{n-\nu-1}\frac{d^n}{dt'^n}f(t')\,dt', \tag{13}$$

for $n-1<\nu<n$, $n\in N$.[1] This subordination approach was used also to find the solution of different telegrapher's equations [29,30,43]. We note that for $\alpha=0$ Eq. (11) reduces to the one analyzed in [44].

As we mentioned above, the telegrapher's process without memory is a random motion with finite velocity. However, this finite velocity property is lost in the fractional generalization. This can be explained by the subordination integral (8) in which the subordination function (9) has a support in $(0,\infty)$ in u variable for any $t>0$, which allows the particle to be at any arbitrary distance at any t with a certain probability [65].

The corresponding mean squared displacement (MSD) for Eq. (11) can be obtained from Eq. (A16). It is given by [30]

$$\langle x^2(t)\rangle=\frac{\Gamma(1+2\rho)(\mathcal{D}_{\alpha}\tau^{\mu})^{\rho}}{\rho^{2\rho}}\times\left(\frac{t}{\tau}\right)^{2\rho\mu}E^{\rho}_{\mu,2\rho\mu+1}\left(-\left[\frac{t}{\tau}\right]^{\mu}\right),\quad \rho=\frac{2}{2-\alpha}, \tag{14}$$

where $E^{\delta}_{\alpha,\beta}(z)$ is the three-parameter Mittag-Leffler function (B11). From Eq. (14) we observe characteristic crossover dynamics, see Eq. (B13),

$$\langle x^2(t)\rangle\sim\begin{cases}t^{\frac{4\mu}{2-\alpha}}, & t\to 0\\ t^{\frac{2\mu}{2-\alpha}}, & t\to\infty\end{cases}. \tag{15}$$

The anomalous diffusion exponent in the short time limit is twice the anomalous diffusion exponent in the long time limit. Here, the specific combination between heterogeneity, represented by the exponent α, and memory, represented by the exponent μ, will determine the nature of the process, see Fig. 1. Thus, if $0<\frac{2\mu}{2-\alpha}<\frac{1}{2}$, i.e., $\mu<\frac{2-\alpha}{4}$ there is a transition from subdiffusion with anomalous diffusion exponent $0<\frac{4\mu}{2-\alpha}<1$ to subdiffusion with anomalous diffusion exponent $0<\frac{2\mu}{2-\alpha}<\frac{1}{2}$. If $\frac{2\mu}{2-\alpha}=\frac{1}{2}$, which means $\mu=\frac{2-\alpha}{4}$, there is a transition from normal diffusion to subdiffusion with anomalous diffusion exponent $1/2$. For $\frac{1}{2}<\frac{2\mu}{2-\alpha}<1$, that is $\frac{2-\alpha}{4}<\mu<\frac{2-\alpha}{2}$, there are characteristic crossover dynamics from superdiffusion to subdiffusion. If $\frac{2\mu}{2-\alpha}=1$ ($\mu=\frac{2-\alpha}{2}$) there is a transition from ballistic motion to normal diffusion.

[1]The Laplace transform, $\hat{f}(s)=\int_0^{\infty}e^{-st}f(t)\,dt$, of the Caputo fractional derivative reads [64]

$$\mathcal{L}\left[\frac{\partial^{\nu}}{\partial t^{\nu}}f(t)\right]=s^{\nu}\hat{f}(s)-\sum_{k=0}^{n-1}s^{\nu-k-1}\left[\lim_{t\to 0}\frac{d^k}{dt^k}f(t)\right].$$

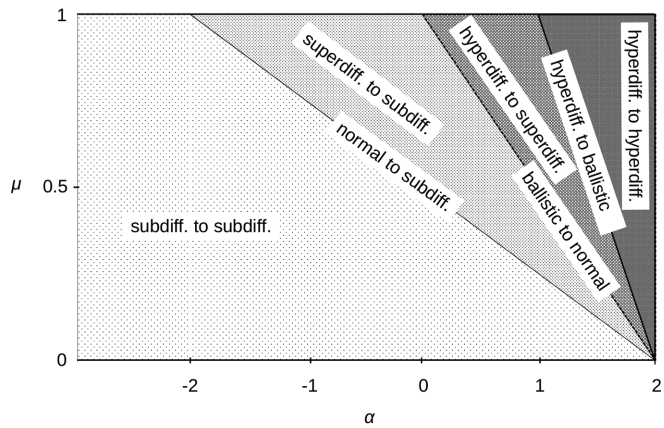

FIG. 1. Characteristic crossover dynamics: interplay between heterogeneity ($\alpha<2$) and memory ($0<\mu<1$).

For $1<\frac{2\mu}{2-\alpha}<2$, i.e., $\frac{2-\alpha}{2}<\mu<2-\alpha$, one observes the transition from hyperdiffusion (anomalous diffusion exponent greater than 2) to superdiffusion, etc.

The MSD for Eq. (12) can be obtained from Eq. (A18). It reads

$$\langle x^2(t)\rangle=\frac{2^{2\rho}\Gamma(\rho)\Gamma(3\rho/2)(\mathcal{D}_{\alpha}\tau^{\mu})^{\rho}}{\rho^{2\rho-1}\Gamma(\rho/2)}\times\left(\frac{t}{\tau}\right)^{2\rho\mu}E^{\rho}_{\mu,2\rho\mu+1}\left(-\left[\frac{t}{\tau}\right]^{\mu}\right). \tag{16}$$

So, it has the same behavior as the MSD (14), only with different prefactors.

III. FIRST-PASSAGE PROPERTIES

Here we will analyze the first-passage time when the process is governed by Eqs. (11) and (12). It is defined as the time required by the particle starting at $x=x_0$ to reach a target located at $x=0$ for the first time. From Eq. (A5) for the PDF, we write the corresponding backward equation for the survival probability $Q(x_0,t)$, which gives the probability that the particle starting at $x=x_0>0$ has not reached the target located at $x=0$ up to time t. Thus, we have

$$\tau^{\mu}\frac{\partial^{2\mu}}{\partial t^{2\mu}}Q(x_0,t)+\frac{\partial^{\mu}}{\partial t^{\mu}}Q(x_0,t)=D(x_0)\frac{\partial^2}{\partial x_0^2}Q(x_0,t)+\left(1-\frac{A}{2}\right)\frac{d\mathcal{D}(x_0)}{dx_0}\frac{\partial}{\partial x}Q(x_0,t), \tag{17}$$

with initial conditions

$$Q(x_0,0)=1,\qquad \left.\frac{\partial Q(x_0,t)}{\partial t}\right|_{t=0}=0, \tag{18}$$

and boundary conditions $Q(0,t)=0$ and $Q(\infty,t)=1$. From the survival probability we can calculate the first-passage time density as

$$\wp(t)=-\frac{d}{dt}Q(x_0,t), \tag{19}$$

i.e.,

$$\wp(s) = 1 - s\hat{q}(x_0, s), \tag{20}$$

in the Laplace space. The hat symbol stands for the Laplace transform. Next, we analyze two cases, $A = 1$ and $A = 0$, which are considered in this paper.

It is to be noted here that the choice of boundary conditions augmenting Eq. (17) is not along the lines, for example, for a run-and-tumble particle under Poissonian tumbling events [66,67]. The reason for this difference is that while the motion of a run-and-tumble particle, described by $P_0(x,t)$, can be decomposed into the motion of individual components, that is, $P_0(x,t) = P_+(x,t) + P_-(x,t)$ [eventually leading to Eq. (1)], no such decomposition exists for FHTEs owing to the non-Markovian nature of tumblings. In other words, the equivalence between studying the individual components $P_\pm(x,t)$ and their sum $P_0(x,t)$ does not go beyond the realm of Poissonian tumblings. As a result, the boundary conditions are specified for the survival probability in the manner discussed above [68].

A. First-passage time for FHTE 1

For the case $A = 1$, the backward equation for the survival probability reads

$$\begin{aligned}\tau^\mu \frac{\partial^{2\mu} Q_1(x_0,t)}{\partial t^{2\mu}} &+ \frac{\partial^\mu Q_1(x_0,t)}{\partial t^\mu} \\ &= \mathcal{D}_\alpha x_0^\alpha \frac{\partial^2 Q_1(x_0,t)}{\partial x_0^2} \\ &\quad + \frac{\mathcal{D}_\alpha \alpha}{2} x_0^{\alpha-1} \frac{\partial Q_1(x_0,t)}{\partial x_0}.\end{aligned} \tag{21}$$

By Laplace transform and by using the initial conditions (18), we have

$$\begin{aligned}\tau^\mu s^{2\mu-1}[s\hat{q}_1(x_0,s) - 1] &+ s^{\mu-1}[s\hat{q}(x_0,s) - 1] \\ &= \mathcal{D}_\alpha x_0^\alpha \frac{\partial^2 \hat{q}_1(x_0,s)}{\partial x_0^2} + \frac{\mathcal{D}_\alpha \alpha}{2} x_0^{\alpha-1} \frac{\partial \hat{q}_1(x_0,s)}{\partial x_0},\end{aligned} \tag{22}$$

so that, rearranging terms

$$\begin{aligned}\frac{\partial^2 \hat{q}_1(x_0,s)}{\partial x_0^2} &+ \frac{\alpha}{2x_0}\frac{\partial \hat{q}_1(x_0,s)}{\partial x_0} - \frac{s^\mu(\tau^\mu s^\mu + 1)}{\mathcal{D}_\alpha x_0^\alpha}\hat{q}_1(x_0,s) \\ &= -\frac{s^{\mu-1}(\tau^\mu s^\mu + 1)}{\mathcal{D}_\alpha x_0^\alpha}.\end{aligned} \tag{23}$$

The homogeneous equation

$$\begin{aligned}\frac{\partial^2 \hat{q}_{1,h}(x_0,s)}{\partial x_0^2} &+ \frac{\alpha}{2x_0}\frac{\partial \hat{q}_{1,h}(x_0,s)}{\partial x_0} \\ &- \frac{s^\mu(\tau^\mu s^\mu + 1)}{\mathcal{D}_\alpha} x_0^{-\alpha}\hat{q}_{1,h}(x_0,s) = 0,\end{aligned} \tag{24}$$

corresponds to the Lommel equation (A14) with

$$\begin{aligned}\bar{\beta} &= \frac{2-\alpha}{4} = \frac{1}{2\rho}, \quad \bar{\alpha} = \frac{2-\alpha}{2} = \frac{1}{\rho}, \quad \bar{\nu} = 1/2, \\ a &= \rho\sqrt{\frac{s^\mu(\tau^\mu s^\mu + 1)}{\mathcal{D}_\alpha}} = \rho R(s).\end{aligned} \tag{25}$$

From here it follows that the solution to the homogeneous equation is

$$\hat{q}_{1,h}(x_0,s) = x_0^{1/2\rho} Z_{1/2}\big(\imath\rho R(s) x_0^{1/\rho}\big), \tag{26}$$

where $Z_{\bar{\nu}}(\imath y) = C_1 I_{\bar{\nu}}(y) + C_2 K_{\bar{\nu}}(y)$, and $I_{\bar{\nu}}(y)$ and $K_{\bar{\nu}}(y)$ are the modified Bessel functions. Since at infinity ($x_0 \to \infty$) $\hat{q}_h(x_0,s)$ should be finite and since $I_{\bar{\nu}}(\infty) \to \infty$ then we should set $C_1 = 0$. Therefore, the solution reads

$$\hat{q}_{1,h}(x_0,s) = C_2 x_0^{1/2\rho} K_{1/2}\big(\rho R(s) x_0^{1/\rho}\big). \tag{27}$$

It is easy to find that the particular solution of Eq. (23) is $\hat{q}_{1,p}(x_0,s) = \frac{1}{s}$, and thus the full solution reads

$$\hat{q}_1(x_0,s) = \frac{1}{s} + \hat{q}_{1,h}(x_0,s). \tag{28}$$

From the property (A17) of the Bessel functions of the third kind, and from the boundary condition $\hat{q}_1(0,s) = 0$, we calculate the constant C_2, and we obtain the solution

$$\hat{q}_1(x_0,s) = \frac{1}{s}\big[1 - \exp\big(-\rho R(s) x_0^{1/\rho}\big)\big]. \tag{29}$$

From the survival probability we calculate the first-passage time density (19), which is given by

$$\hat{\wp}_1(s) = \exp\big(-\rho R(s) x_0^{1/\rho}\big). \tag{30}$$

In the long time limit $s \to 0$, by using the Tauberian theorem (see Appendix C), we find

$$\frac{1}{s}\hat{\wp}_1(s) \sim \frac{1}{s}\big[1 - \rho R(s) x_0^{1/\rho}\big], \tag{31}$$

where $R(s) = \sqrt{\frac{s^\mu(\tau^\mu s^\mu+1)}{\mathcal{D}_\alpha}} \sim \frac{s^{\mu/2}}{\sqrt{\mathcal{D}_\alpha}}$. From here it follows

$$\int_0^t \wp_1(t')\,dt' \overset{t\to\infty}{\simeq} 1 - \rho\frac{x_0^{1/\rho}}{\sqrt{\mathcal{D}_\alpha}}\frac{t^{-\mu/2}}{\Gamma(1-\mu/2)}. \tag{32}$$

As a result,

$$\wp_1(t) \overset{t\to\infty}{\simeq} \rho\frac{(\mu/2)x_0^{1/\rho}}{\sqrt{\mathcal{D}_\alpha}}\frac{t^{-\mu/2-1}}{\Gamma(1-\mu/2)} \sim \frac{1}{t^{1+\frac{\mu}{2}}}, \tag{33}$$

which is a generalization of the first-passage time density for the Brownian motion, $\sim t^{-3/2}$, and is recovered for $\mu = 1$ [69].

As a special case for $\tau = 0$, we obtain that the first-passage time density for the standard fractional heterogeneous diffusion process obeys the scaling

$$\wp_1(t) = \mathcal{L}^{-1}\left[\exp\left(-\frac{\rho x_0^{1/\rho}}{\sqrt{\mathcal{D}_\alpha}}s^{\mu/2}\right)\right] = \frac{1}{t}f_{\rho,\mu}\left(\frac{\rho x_0^{1/\rho}}{\sqrt{\mathcal{D}_\alpha t^\mu}}\right), \tag{34}$$

where

$$f_{\alpha,\mu}(z) = H_{1,1}^{1,0}\left[z\left|\begin{matrix}(0,\mu/2)\\(0,1)\end{matrix}\right.\right], \tag{35}$$

and $H_{p,q}^{m,n}(z)$ is the Fox H-function (B1), and we applied relations (B8) and (B10). As a special case with $\mu = 1$, the first-passage time density reduces to

$$\wp_1(t) = \frac{1}{t}H_{1,1}^{1,0}\left[\frac{\rho x_0^{1/\rho}}{\sqrt{\mathcal{D}_\alpha t}}\left|\begin{matrix}(0,1/2)\\(0,1)\end{matrix}\right.\right] = \frac{\rho x_0^{1/\rho}}{\sqrt{4\pi\mathcal{D}_\alpha t^3}}e^{-\frac{\rho^2 x_0^{2/\rho}}{4\mathcal{D}_\alpha t}}, \tag{36}$$

which for $\rho = 1$ ($\alpha = 0$) yields the Lévy-Smirnov distribution, see, for example, Ref. [70]

$$\wp_1(t) = \frac{|x_0|}{\sqrt{4\pi\mathcal{D}_0 t^3}} e^{-\frac{x_0^2}{4\mathcal{D}_0 t}}. \quad (37)$$

B. First-passage time for FHTE 2

For the case $A = 0$, the backward equation for the survival probability reads

$$\tau^\mu \frac{\partial^{2\mu} Q_2(x_0,t)}{\partial t^{2\mu}} + \frac{\partial^\mu Q_2(x_0,t)}{\partial t^\mu} = \mathcal{D}_\alpha x_0^\alpha \frac{\partial^2 Q_2(x_0,t)}{\partial x_0^2} + \mathcal{D}_\alpha \alpha x_0^{\alpha-1} \frac{\partial Q_2(x_0,t)}{\partial x_0}. \quad (38)$$

By Laplace transform, we have

$$\tau^\mu s^{2\mu-1}[s\hat{q}_2(x_0,s) - 1] + s^{\mu-1}[s\hat{q}_2(x_0,s) - 1] = \mathcal{D}_\alpha x_0^\alpha \frac{\partial^2 \hat{q}_2(x_0,s)}{\partial x_0^2} + \mathcal{D}_\alpha \alpha x_0^{\alpha-1} \frac{\partial \hat{q}_2(x_0,s)}{\partial x_0}, \quad (39)$$

where we use the initial conditions (18). Therefore, we have the following equation:

$$\frac{\partial^2 \hat{q}_2(x_0,s)}{\partial x_0^2} + \frac{\alpha}{x_0}\frac{\partial \hat{q}_2(x_0,s)}{\partial x_0} - \frac{s^\mu(\tau^\mu s^\mu + 1)}{\mathcal{D}_\alpha x_0^\alpha}\hat{q}_2(x_0,s) = -\frac{s^{\mu-1}(\tau^\mu s^\mu + 1)}{\mathcal{D}_\alpha x_0^\alpha}. \quad (40)$$

The homogeneous part of this equation is again a Lommel-type equation (A14), while the particular solution equals $1/s$. Therefore, the final solution for the survival probability in Laplace space is

$$\hat{q}_2(x_0,s) = \frac{1}{s}\left[1 - \frac{\rho^{\bar{\nu}}}{\Gamma(\bar{\nu})}\frac{R^{\bar{\nu}}(s)}{2^{\bar{\nu}-1}} x_0^{\bar{\nu}/\rho} K_{\bar{\nu}}\left(\rho R(s) x_0^{1/\rho}\right)\right], \quad (41)$$

where $\bar{\nu} = \frac{1-\alpha}{2-\alpha}$. Thus, the first-passage time density becomes

$$\hat{\wp}_2(s) = \frac{\rho^{\bar{\nu}}}{\Gamma(\bar{\nu})}\frac{R^{\bar{\nu}}(s)}{2^{\bar{\nu}-1}} x_0^{\bar{\nu}/\rho} K_{\bar{\nu}}\left(\rho R(s) x_0^{1/\rho}\right). \quad (42)$$

From here, for the long time limit we find

$$\wp_2(t) \sim \frac{\rho^{2\bar{\nu}} x_0^{2\bar{\nu}/\rho}}{2^{2\bar{\nu}}\mathcal{D}_\alpha^{\bar{\nu}}}\frac{(-\mu\bar{\nu})\Gamma(-\bar{\nu})}{\Gamma(\bar{\nu})}\frac{t^{-\mu\bar{\nu}-1}}{\Gamma(1-\mu\bar{\nu})} \sim \frac{1}{t^{1+\mu\frac{1-\alpha}{2-\alpha}}}. \quad (43)$$

This is another generalization of the first-passage time density for the Brownian motion, $\sim t^{-3/2}$, which is recovered for $\alpha = 0$, $\mu = 1$. Note that the temporal scaling of the first passage time density $\wp_2(t)$ reduces to (33) for $\alpha = 0$.

For $\tau = 0$, we obtain that the first-passage time density obeys the scaling

$$\wp_2(t) = \frac{2}{\Gamma(\bar{\nu})}\mathcal{L}^{-1}\left[\left(\frac{\rho x_0^{1/\rho}}{\sqrt{4\mathcal{D}_\alpha}} s^{\mu/2}\right)^{\bar{\nu}} K_{\bar{\nu}}\left(\frac{\rho x_0^{1/\rho}}{\sqrt{\mathcal{D}_\alpha}} s^{\mu/2}\right)\right] = \frac{1}{t} g_{\alpha,\mu}\left(\frac{\rho x_0^{1/\rho}}{\sqrt{4\mathcal{D}_\alpha t^\mu}}\right), \quad (44)$$

where

$$g_{\alpha,\mu}(z) = \frac{1}{2\Gamma(\bar{\nu})} H_{1,2}^{2,0}\left[z\middle|\begin{matrix}(0,\mu/2)\\(\bar{\nu},1/2),(0,1/2)\end{matrix}\right]. \quad (45)$$

Here, we used relations (B6) and (B9), and the Laplace transform formula (B10). For $\mu = 1$ it becomes

$$\wp_2(t) = \frac{1}{\Gamma(\bar{\nu})t} H_{0,1}^{1,0}\left[\frac{\rho^2 x_0^{2/\rho}}{4\mathcal{D}_\alpha t}\middle|(\bar{\nu},1)\right] = \frac{\rho^{2\bar{\nu}} x_0^{2\bar{\nu}/\rho}}{\Gamma(\bar{\nu})(4\mathcal{D}_\alpha)^{\bar{\nu}} t^{\bar{\nu}+1}} e^{-\frac{\rho^2 x_0^{2/\rho}}{4\mathcal{D}_\alpha t}}, \quad (46)$$

where we applied the symmetry property of the Fox H-function and relations (B4) and (B8). We note that for $\alpha = 0$ ($\rho = 1$), both results (34) and (44) reduce to

$$\wp_{1,2}(t) = \frac{1}{t} H_{1,1}^{1,0}\left[\frac{x_0}{\sqrt{\mathcal{D}_\alpha t^\mu}}\middle|\begin{matrix}(0,\mu/2)\\(0,1)\end{matrix}\right], \quad (47)$$

as it should be. Note that the presence of heterogeneity in the fractional telegrapher's equation does not modify the fact that the mean first passage time to $x = 0$ is still infinite.

IV. FHTP UNDER RESETTING

As it was already mentioned, the MFPT of the FHTP diverges. Here, we will show how with stochastic resetting of the particle to its starting point x_0 the MFPT becomes finite and can be optimized with respect to the resetting rate.

Thus, we consider both FHTPs in the presence of stochastic resetting. For Poissonian resetting, we use the renewal equation for the PDF [71–73], i.e.,

$$P_r(x,t) = e^{-rt}P(x,t) + \int_0^t re^{-rt'}P(x,t')\,dt'. \quad (48)$$

The first term from the right-hand side of the equation gives the probability that there was not a resetting event up to time t, while in the second term the probability that there were many resetting events up to time t. From the Laplace transform of the renewal equation, one finds

$$\hat{P}_r(x,s) = \frac{s+r}{s}\hat{P}(x,s+r), \quad (49)$$

from where in the long time limit the nonequilibrium stationary state (NESS) is obtained, i.e.,

$$P^s(x) = \lim_{t\to\infty} P_r(x,t) = \lim_{s\to 0} s\hat{P}_r(x,s) = r\hat{P}(x,r). \quad (50)$$

From the renewal equation, one can also find the survival probability $Q_r(x_0,t)$ in the presence of resetting if one knows the survival probability $Q(x_0,t)$ in the absence of resetting. In Laplace space it reads [35,74,75]

$$\hat{q}_r(x_0,s) = \frac{\hat{q}(x_0,s+r)}{1 - r\hat{q}(x_0,s+r)}. \quad (51)$$

From here, we find the MFPT

$$\langle T_r\rangle = -\int_0^\infty t\frac{\partial Q_r(x_0,t)}{\partial t}\,dt = \hat{q}_r(x_0,s=0). \quad (52)$$

Next, we will analyze the MFPT for both cases of the fractional heterogeneous telegrapher's process.

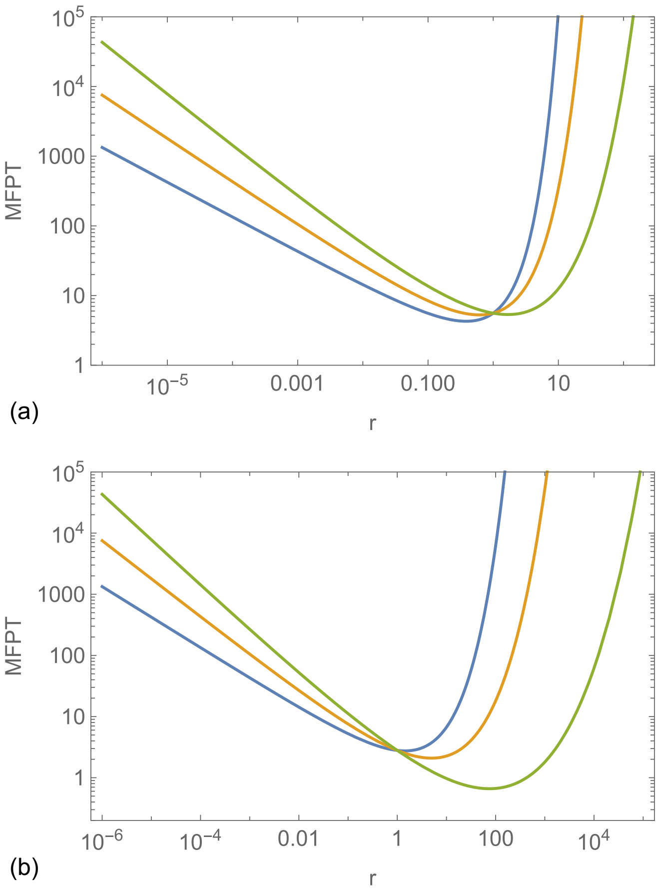

FIG. 2. MFPT (54) for (a) $\tau = 1$ and (b) $\tau = 0$. We set $x_0 = 1$, $\alpha = 1/2$, $\mu = 1$ (blue line), $\mu = 3/4$ (orange line), and $\mu = 1/2$ (green line).

A. FHTE 1 under resetting

Therefore, for the FHTE 1, we find

$$\hat{q}_{r,1}(x_0, s) = \frac{\hat{q}_1(x_0, s+r)}{1 - r\hat{q}_1(x_0, s+r)} = \frac{1 - \exp\left(-\rho R(s+r)x_0^{1/\rho}\right)}{s + r\exp\left(-\rho R(s+r)x_0^{1/\rho}\right)}, \tag{53}$$

from where it follows

$$\langle T_r\rangle_1 = \frac{1}{r}\left[\exp\left(\rho R(r)x_0^{1/\rho}\right) - 1\right]. \tag{54}$$

For all $\alpha < 2$ the MFPT is infinite for $r = 0$ and $r = \infty$, and thus there is an optimal resetting rate r_* for which the MFPT has its minimum. This optimal resetting rate can be obtained from

$$\frac{\partial}{\partial r}\langle T(x_0)\rangle_1\bigg|_{r=r_*} = 0. \tag{55}$$

Thus, we obtain

$$1 - e^{-\xi} = r_*\frac{d\xi}{dr_*}, \quad \xi = R(r_*)x_0^{1/\rho}. \tag{56}$$

A graphical representation of the MFPT (54) is given in Fig. 2. From the figures, one can conclude that for $r = 1$, keeping α (or equivalently ρ) fixed, all curves intersect at

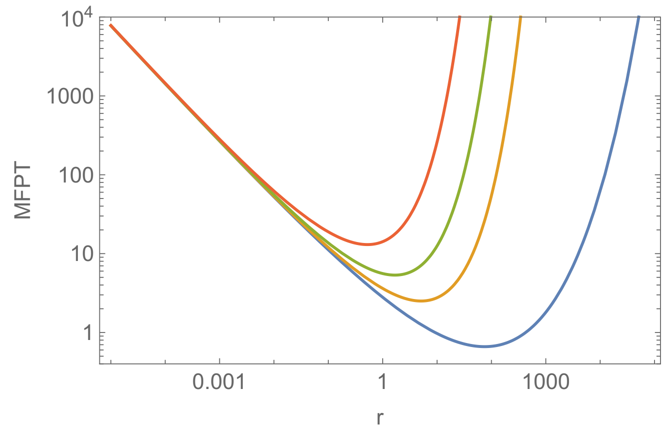

FIG. 3. MFPT (54) for $\tau = 0$ (blue line), $\tau = 0.1$ (orange line), $\tau = 1$ (green line), and $\tau = 10$ (red line). We set $x_0 = 1$, $\alpha = 1/2$, and $\mu = 1/2$.

$r = 1$, i.e., the MFPTs are the same. The minima of MFPT monotonically increase with decreasing μ for $\tau = 1$, while an opposite behavior is observed for $\tau = 0$. The optimal resetting rate r_* for the telegrapher's process increases with the decrease of μ, while for the diffusion process, it is opposite—the optimal resetting rate increases with the increase of μ.

In Fig. 3, it is observed that the optimal resetting rate is decreasing with increasing τ, while the minima of the MFPT increase with an increase of τ.

B. FHTE 2 under resetting

For the FHTE 2, we have

$$\hat{q}_{r,2}(x_0, s) = \frac{\hat{q}_2(x_0, s+r)}{1 - r\hat{q}_2(x_0, s+r)} = \frac{1 - \frac{\rho^{\bar{\nu}}}{\Gamma(\bar{\nu})}\frac{R^{\bar{\nu}}(s+r)}{2^{\bar{\nu}-1}}x_0^{\bar{\nu}/\rho}K_{\bar{\nu}}\left(\rho R(s+r)x_0^{1/\rho}\right)}{s + r\frac{\rho^{\bar{\nu}}}{\Gamma(\bar{\nu})}\frac{R^{\bar{\nu}}(s+r)}{2^{\bar{\nu}-1}}x_0^{\bar{\nu}/\rho}K_{\bar{\nu}}\left(\rho R(s+r)x_0^{1/\rho}\right)}, \tag{57}$$

and thus

$$\langle T_r\rangle_2 = \frac{1}{r}\left[\frac{2^{\bar{\nu}-1}\Gamma(\bar{\nu})}{\rho^{\bar{\nu}}R^{\bar{\nu}}(r)x_0^{\bar{\nu}/\rho}K_{\bar{\nu}}\left(\rho R(r)x_0^{1/\rho}\right)} - 1\right]. \tag{58}$$

From here, one concludes that in order the MFPT is finite then $\bar{\nu} = \frac{1-\alpha}{2-\alpha} > 0$, i.e., $\alpha < 1$. Since this condition is always fulfilled, the MFPT is always finite. For $\tau = 0$ we recover the result obtained in Ref. [35].

A graphical representation of the MFPT (58) is given in Fig 4. From the figure, one can conclude the same behavior of the MFPT with the resetting rate r, as in the previous case. The MFPTs are the same at $r = 1$ for fixed α (or equivalently ρ), the minima of MFPT monotonically increase with decreasing μ for $\tau = 1$, while an opposite behavior for $\tau = 0$ is observed. The optimal resetting rate r_* for the telegrapher's process increases with the decrease of μ, while for the diffusion process, the optimal resetting rate increases with the increase of μ.

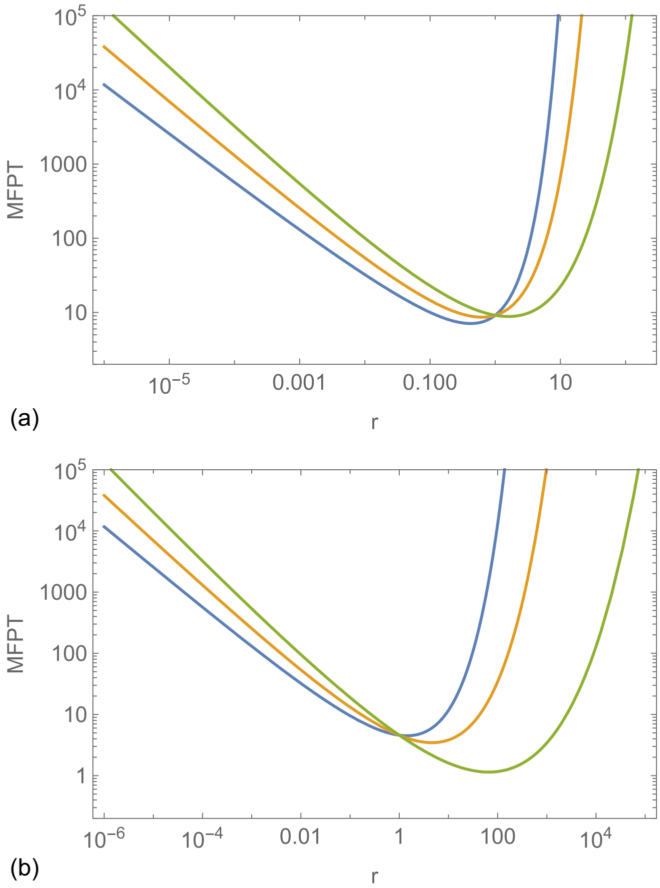

FIG. 4. MFPT ([58]) for (a) $\tau = 1$ and (b) $\tau = 0$. We set $x_0 = 1$, $\alpha = 1/2$, $\mu = 1$ (blue line), $\mu = 3/4$ (orange line), and $\mu = 1/2$ (green line).

In Fig. 5, it is also observed, as in the previous case, that the optimal resetting rate decreases with increasing τ, while the minima of the MFPT increase with the increase of τ.

A graphical representation of the comparison between MFPTs ([54]) and ([58]) is given in Fig. 6. One can conclude that $\langle T_r \rangle_1 < \langle T_r \rangle_2$.

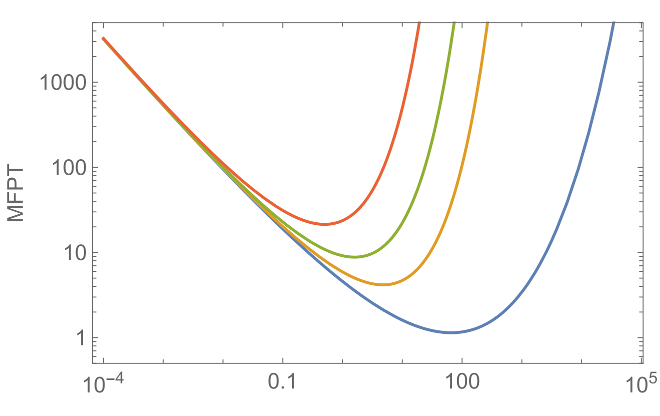

FIG. 5. MFPT ([58]) for $\tau = 0$ (blue line), $\tau = 0.1$ (orange line), $\tau = 1$ (green line), and $\tau = 10$ (red line). We set $x_0 = 1$, $\alpha = 1/2$ and $\mu = 1/2$.

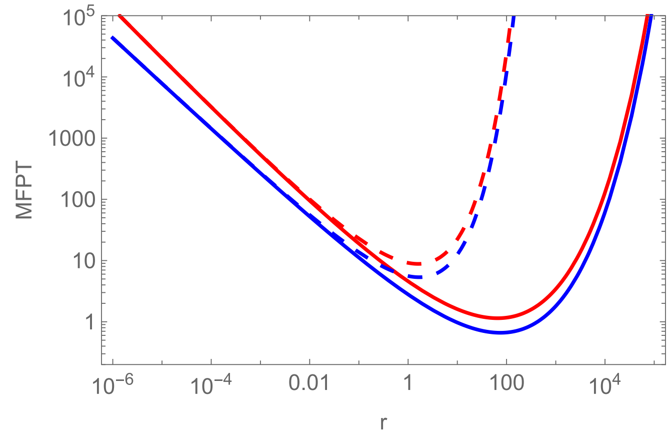

FIG. 6. Comparison between MFPTs ([54]) (blue lines) and ([58]) (red lines) for $\tau = 1$ (dashed lines) and $\tau = 0$ (solid lines). We set $x_0 = 1$, $\alpha = 1/2$, and $\mu = 1/2$.

V. SUMMARY

In this paper, we considered two different forms of FHTEs and found their solutions. The corresponding MSDs share the same behavior in time and are presented through the three-parameter Mittag-Leffler function. Both models show anomalous diffusive behavior where the anomalous diffusion exponent in the short time limit is twice the anomalous diffusion exponent in the long time limit, which is a signature of characteristic crossover dynamics. The first passage times in the long time limit have different temporal power-law behavior. We also showed that in the presence of stochastic resetting, the MFPTs for both models become finite and that there are optimal resetting rates at which the MFPTs are minimal. The dependence of the optimal reset rate on the parameters α and μ are qualitatively the same for both models, while for $\tau = 0$ the behavior is the opposite.

The influence of different resetting mechanisms on the search strategy, such as time-dependent [76], noninstantaneous [77,78], partial resetting [79,80], resetting in an interval [81,82], discrete space-time resetting models [83], and stochastic resetting to multiple [84] and random positions [85], could be of interest for future research. The more generalized model of the heterogeneous telegrapher's equation with general memory kernel [38] in the presence of resetting, as well as with time-dependent diffusion coefficient as in the scaled Brownian motion [73], we leave for future investigation.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: FROM STANDARD HTE TO FHTE AND GENERAL SOLUTION

By Laplace transform to (7), we have

$$(\tau s+1)[s\hat{P}(x,s)-\delta(x)] = \frac{\partial}{\partial x}\left\{D(x)^{1-A/2}\frac{\partial}{\partial x}[D(x)^{A/2}\hat{P}(x,s)]\right\}. \tag{A1}$$

We introduce the substitution $s \to \tau^{\mu-1}s^{\mu}$ to obtain

$$(\tau^{\mu}s^{\mu}+1)[\tau^{\mu-1}s^{\mu}\hat{P}(x,\tau^{\mu-1}s^{\mu})-\delta(x)] = \frac{\partial}{\partial x}\left\{D(x)^{1-A/2}\frac{\partial}{\partial x}[D(x)^{A/2}\hat{P}(x,\tau^{\mu-1}s^{\mu})]\right\}. \tag{A2}$$

We define a new PDF as follows:

$$\hat{P}_{s}(x,s)=\tau^{\mu-1}s^{\mu-1}\hat{P}(x,\tau^{\mu-1}s^{\mu}), \tag{A3}$$

from where it follows

$$s^{\mu-1}(\tau^{\mu}s^{\mu}+1)[s\hat{P}_{s}(x,s)-\delta(x)] = \frac{\partial}{\partial x}\left\{\mathcal{D}(x)^{1-A/2}\frac{\partial}{\partial x}[\mathcal{D}(x)^{A/2}\hat{P}_{s}(x,s)]\right\}, \tag{A4}$$

where $\mathcal{D}(x)=\tau^{1-\mu}D(x)$. The inverse Laplace transform of (A4) yields the following FHTE:

$$\tau^{\mu}\frac{\partial^{2\mu}}{\partial t^{2\mu}}P_{s}(x,t)+\frac{\partial^{\mu}}{\partial t^{\mu}}P_{s}(x,t) = \frac{\partial}{\partial x}\left\{\mathcal{D}(x)^{1-A/2}\frac{\partial}{\partial x}[\mathcal{D}(x)^{A/2}P_{s}(x,t)]\right\}. \tag{A5}$$

From Eq. (A3), we have

$$\hat{P}_{s}(x,s)=\int_{0}^{\infty}\tau^{\mu-1}s^{\mu-1}e^{-u\tau^{\mu-1}s^{\mu}}P(x,u)\,du. \tag{A6}$$

The inverse Laplace transform yields

$$P_{s}(x,t)=\int_{0}^{\infty}P(x,u)h(u,t)\,du, \tag{A7}$$

with

$$\begin{aligned} h(u,t) &= \mathcal{L}^{-1}[\tau^{\mu-1}s^{\mu-1}e^{-u\tau^{\mu-1}s^{\mu}}] \\ &= \frac{1}{\tau(t/\tau)^{\mu}}H_{1,1}^{1,0}\left[\frac{u/\tau}{(t/\tau)^{\mu}}\middle|\begin{matrix}(1-\mu,\mu)\\(0,1)\end{matrix}\right] \\ &= \frac{t/\tau}{\mu u(u/\tau)^{1/\mu}}L_{\mu}\left(\frac{t/\tau}{(u/\tau)^{1/\mu}}\right). \end{aligned} \tag{A8}$$

Here, $H_{p,q}^{m,n}(z)$ is the Fox H-function (B1) and $L_{\alpha}(z)$ is the $L_{\alpha}(z)$ is the one-sided Lévy stable PDF (B7). This procedure is known as the subordination approach, i.e., the integral (A7) is called subordination integral, while the function $h(u,t)$ (A8)—subordination function.

Let us now solve Eq. (A5) for $\mathcal{D}(x)=\mathcal{D}_{\alpha}|x|^{\alpha}$. Thus, we have

$$s^{\mu-1}(\tau^{\mu}s^{\mu}+1)[s\hat{P}_{s}(x,s)-\delta(x)] = \mathcal{D}_{\alpha}\frac{\partial}{\partial x}\left\{|x|^{\frac{(2-A)\alpha}{2}}\frac{\partial}{\partial x}\left[|x|^{\frac{A\alpha}{2}}\hat{P}_{s}(x,s)\right]\right\}. \tag{A9}$$

Performing differentiation with respect to x, one finds

$$\begin{aligned} s\hat{P}_{s}(x,s)-\delta(x) &= \frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)}\Big[A\alpha\delta(x)|x|^{\alpha-1}\hat{P}_{s}(x,s) \\ &\quad+(2\theta(x)-1)\frac{(A+2)\alpha}{2}|x|^{\alpha-1}\frac{\partial}{\partial x}\hat{P}_{s}(x,s) \\ &\quad+\frac{A(\alpha-1)\alpha}{2}|x|^{\alpha-2}\hat{P}_{s}(x,s)+|x|^{\alpha}\frac{\partial^{2}}{\partial x^{2}}\hat{P}_{s}(x,s)\Big]. \end{aligned} \tag{A10}$$

Taking into account that the Fokker-Planck equation is symmetrical with respect to inversion $x\to -x$, we can consider the solution for the non-negative x, when $x=|x|$ and then extend it symmetrically for the entire x axis. Therefore, using the variable change $\hat{P}(|x|,s)=\mathcal{C}(s)\hat{f}(|x|,s)=\mathcal{C}(s)\hat{f}(y,s)$, where $\mathcal{C}(s)$ is a function of s, we transform Eq. (A10) to

$$\begin{aligned} s\hat{f}(y,s)-\frac{\delta(x)}{\mathcal{C}(s)} &= \frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)}\frac{A(\alpha-1)\alpha}{2}y^{\alpha-2}\hat{f}(y,s) \\ &\quad+\frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)}A\alpha y^{\alpha-1}\delta(x)\hat{f}(y,s) \\ &\quad+\frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)}\frac{(A+2)\alpha}{2}y^{\alpha-1}\frac{\partial}{\partial y}\hat{f}(y,s) \\ &\quad+2\frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)}y^{\alpha}\delta(x)\frac{\partial}{\partial y}\hat{f}(y,s) \\ &\quad+\frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)}y^{\alpha}\frac{\partial^{2}}{\partial y^{2}}\hat{f}(y,s). \end{aligned} \tag{A11}$$

Separating terms with $\delta(x)$, we obtain two independent equations:

$$\frac{\partial^{2}}{\partial y^{2}}\hat{f}(y,s)+\frac{(A+2)\alpha/2}{y}\frac{\partial}{\partial y}\hat{f}(y,s) +\left[-\frac{s^{\mu}(\tau^{\mu}s^{\mu}+1)/\mathcal{D}_{\alpha}}{y^{\alpha}}+\frac{A(\alpha-1)\alpha/2}{y^{2}}\right]\hat{f}(y,s)=0, \tag{A12}$$

$$-\frac{1}{\mathcal{C}(s)}=\frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)} \times\left[A\alpha y^{\alpha-1}\hat{f}(y,s)+2y^{\alpha}\frac{\partial}{\partial y}\hat{f}(y,s)\right]\Big|_{y=0}. \tag{A13}$$

Equation (A12) is the Lommel-type differential equation:

$$z''(y)+\frac{1-2\bar{\beta}}{y}z'(y) +\left[\left(a\bar{\alpha}y^{\bar{\alpha}-1}\right)^{2}+\frac{\bar{\beta}^{2}-\bar{\nu}^{2}\bar{\alpha}^{2}}{y^{2}}\right]z(y)=0, \tag{A14}$$

where a, $\bar{\nu}$, $\bar{\alpha}$, and $\bar{\beta}$ are parameters, while primes for z denote derivatives with respect to y. The solution of Eq. (A14) is

$$z(y) = y^{\bar{\beta}} Z_{\bar{\nu}}\big(\imath a y^{\bar{\alpha}}\big),$$

where $Z_{\bar{\nu}}(y) = C_1 J_{\bar{\nu}}(y) + C_2 Y_{\bar{\nu}}(y)$, and $J_{\bar{\nu}}(y)$ and $Y_{\bar{\nu}}(y)$ are the Bessel functions of first and second kind. The boundary conditions at infinity are equal to zero. Therefore, the solution reads

$$z(y) = y^{\bar{\beta}} K_{\bar{\nu}}(a y^{\bar{\alpha}}),$$

where $K_{\bar{\nu}}(y)$ is the modified Bessel function (of the third kind). Here, we also find the relations

$$a = \frac{2}{2-\alpha}\sqrt{\frac{s^{\mu}(\tau^{\mu}s^{\mu}+1)}{\mathcal{D}_{\alpha}}}, \quad \bar{\alpha} = \frac{2-\alpha}{2}, \quad \bar{\beta} = \frac{2-(A+2)\alpha}{4},$$
$$\bar{\nu} = \frac{[2-(A+2)\alpha]^2 - 8A(\alpha-1)\alpha}{8(2-\alpha)}. \tag{A15}$$

Inserting the obtained solution $\hat{f}(y,s)$ in Eq. (A13), we find $\mathcal{C}(s)$.

1. Case with $A = 1$

For this case, we have $\bar{\beta} = \frac{2-3\alpha}{4}$ and $\bar{\nu} = 1/2$, i.e.,

$$\hat{P}_{\mathrm{s}}(x,s) = \left(\frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)}\right)^{-3/4} \frac{s^{-1/4}}{\sqrt{(2-\alpha)\pi}}$$
$$\times |x|^{\frac{2-3\alpha}{4}} K_{\frac{1}{2}}\left(\frac{2}{2-\alpha}\sqrt{\frac{s^{\mu}(\tau^{\mu}s^{\mu}+1)}{\mathcal{D}_{\alpha}}}|x|^{\frac{2-\alpha}{2}}\right)$$
$$= \frac{|x|^{-\alpha/2}}{2} s^{-1}\sqrt{\frac{s^{\mu}(\tau^{\mu}s^{\mu}+1)}{\mathcal{D}_{\alpha}}}$$
$$\times \exp\left(-\frac{2}{2-\alpha}\sqrt{\frac{s^{\mu}(\tau^{\mu}s^{\mu}+1)}{\mathcal{D}_{\alpha}}}|x|^{(2-\alpha)/2}\right), \tag{A16}$$

where we use the relation between the modified Bessel function (of the third kind) and exponential function,

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}. \tag{A17}$$

2. Case with $A = 0$

For this case, we have $\bar{\beta} = \frac{1-\alpha}{2}$ and $\bar{\nu} = \frac{1-\alpha}{2-\alpha}$, which yields

$$\hat{P}_{\mathrm{s}}(x,s) = \left(\frac{\mathcal{D}_{\alpha}}{s^{\mu-1}(\tau^{\mu}s^{\mu}+1)}\right)^{-(3-\alpha)/[2(2-\alpha)]}$$
$$\times \frac{s^{-(1-\alpha)/[2(2-\alpha)]}}{\Gamma\left(\frac{1}{2-\alpha}\right)(2-\alpha)^{1/(2-\alpha)}}$$
$$\times |x|^{\frac{1-\alpha}{2}} K_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2}{2-\alpha}\sqrt{\frac{s^{\mu}(\tau^{\mu}s^{\mu}+1)}{\mathcal{D}_{\alpha}}}|x|^{\frac{2-\alpha}{2}}\right). \tag{A18}$$

APPENDIX B: FOX H-FUNCTION AND MITTAG-LEFFLER FUNCTIONS

The Fox H-function is defined by means of the following Mellin-Barnes integral [86–88]:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right]$$
$$= H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = \frac{1}{2\pi\imath}\int_{\Omega}\theta(s)z^{-s}\,ds, \tag{B1}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}, \tag{B2}$$

$0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q$, $a_i, b_j \in \mathrm{C}$, $A_i, B_j \in \mathrm{R}^{+}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$. Contour integration Ω starts at $c - \imath\infty$ and finishes at $c + \imath\infty$, separating the poles of the function $\Gamma(b_j + B_j s)$, $j = 1, \ldots, m$ with those of the function $\Gamma(1 - a_i - A_i s)$, $i = 1, \ldots, n$.

(Symmetric property) The Fox H-function is symmetric in the following pairs $(a_1, A_1), \ldots, (a_n, A_n)$, as well as in the pairs $(a_{n+1}, A_{n+1}), \ldots, (a_p, A_p)$. The Fox H-function is symmetric also in the pairs $(b_1, B_1), \ldots, (b_m, B_m)$, as well as in the pairs $(b_{m+1}, B_{m+1}), \ldots, (b_q, B_q)$.

The following reduction formulas for the Fox H-function for $n \geqslant 1$, $q > m$, are valid:

$$H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(a_1,A_1)\end{matrix}\right]$$
$$= H_{p-1,q-1}^{m,n-1}\left[z\middle|\begin{matrix}(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_{q-1},B_{q-1})\end{matrix}\right], \tag{B3}$$

$$H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_1,A_1),\ldots,(a_{p-1},A_{p-1}),(b_1,B_1)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right]$$
$$= H_{p-1,q-1}^{m-1,n}\left[z\middle|\begin{matrix}(a_1,A_1),\ldots,(a_{p-1},A_{p-1})\\(b_2,B_2),\ldots,(b_q,B_q)\end{matrix}\right]. \tag{B4}$$

Moreover, for $\delta > 0$, the following formula holds true:

$$H_{p,q}^{m,n}\left[z^{\delta}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = \frac{1}{\delta}H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,A_p/\delta)\\(b_q,B_q/\delta)\end{matrix}\right], \tag{B5}$$

as well as

$$z^{\sigma}H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right] = H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_p+\sigma A_p,A_p)\\(b_q+\sigma B_q,B_q)\end{matrix}\right]. \tag{B6}$$

The one-sided Levy stable PDF $L_{\alpha}(z)$ is defined by the following Laplace transform [88]:

$$\mathcal{L}[L_{\alpha}(t)] = e^{-s^{\alpha}}. \tag{B7}$$

The Fox H-function is related to the exponential as follows [88]:

$$H_{0,1}^{1,0}\left[z\middle|(\alpha,1)\right] = z^{\alpha}e^{-z}, \tag{B8}$$

and with the modified Bessel function (of the third kind) as [88]

$$H_{0,2}^{2,0}\left[z\middle|\left(\tfrac{\nu}{2},1\right),\left(-\tfrac{\nu}{2},1\right)\right] = 2K_{\nu}(2z^{1/2}). \tag{B9}$$

The inverse Laplace transform of the Fox H-function reads [88]

$$\mathcal{L}^{-1}\left[s^{-\rho}H_{p,q}^{m,n}\left[as^{\sigma}\middle|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right]\right] = t^{\rho-1}H_{p+1,q}^{m,n}\left[\frac{a}{t^{\sigma}}\middle|\begin{matrix}(a_p,A_p),(\rho,\sigma)\\(b_q,B_q)\end{matrix}\right], \tag{B10}$$

where $\rho, a, s \in C$, $\Re(s) > 0$, $\sigma > 0$, $\Re(\rho) + \sigma\max_{1\leqslant i\leqslant n}[\frac{1}{A_i} - \frac{\Re(a_i)}{A_i}] > 0$, $|\arg(a)| < \frac{\pi\theta}{1}$, $\theta = \alpha - \sigma$.

The three-parameter Mittag-Leffler function is defined by [89]

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}, \tag{B11}$$

with $\beta, \gamma, z \in C$, $\Re(\alpha) > 0$, $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$ is the Pochhammer symbol. For $\gamma = 1$, it reduces to the two-parameter Mittag-Leffler function, $E_{\alpha,\beta}(z) = E_{\alpha,\beta}^{\gamma}(z)$ and for $\beta = \gamma = 1$—to the one-parameter Mittag-Leffler function, $E_{\alpha}(z) = E_{\alpha,1}(z) = E_{\alpha,1}^{1}(z)$.

The Laplace transform of the three-parameter Mittag-Leffler function reads [89]

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}^{\gamma}(\pm\lambda t^{\alpha})\right] = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}\mp\lambda)^{\gamma}}, \tag{B12}$$

where $|\lambda/s^{\alpha}| < 1$.

The three-parameter Mittag-Leffler function has the following asymptotic behavior [90]:

$$E_{\alpha,\beta}^{\gamma}(-\lambda t^{\alpha}) \sim \frac{1}{\lambda^{\gamma}}\frac{t^{-\alpha\gamma}}{\Gamma(\beta-\alpha\gamma)}, \quad t \gg 1. \tag{B13}$$

APPENDIX C: TAUBERIAN THEOREMS

The Tauberian theorem states that if the asymptotic behavior of a given function $r(t)$ for $t \to \infty$ is given by [91]

$$r(t) \sim t^{-\alpha}, \quad \alpha > 0, \tag{C1}$$

then, the corresponding Laplace pair $\hat{r}(s) = \mathcal{L}[r(t)]$ has the behavior

$$\hat{r}(s) \sim \Gamma(1-\alpha)s^{\alpha-1}, \tag{C2}$$

for $s \to 0$. The theorem also works in the opposite direction, ensuring that $r(t)$ is the non-negative and monotone function at infinity.

The Tauberian theorem can be formulated in the form of the Hardy-Littlewood theorem, which states that, if the Laplace-Stieltjes transform of a given non-decreasing function F with $F(0) = 0$, defined by Stieltjes integral

$$\omega(s) = \int_0^{\infty} e^{-st}\, dF(t), \tag{C3}$$

has asymptotic behavior

$$\omega(s) \sim Cs^{-\nu}, \quad s \to \infty \quad (s \to 0), \tag{C4}$$

where $\nu \geqslant 0$ and C are real numbers, then the function F has asymptotic behavior

$$F(t) \sim \frac{C}{\Gamma(\nu+1)}t^{\nu}, \quad t \to 0 \quad (t \to \infty). \tag{C5}$$

[1] I. M. Sokolov, Soft Matter **8**, 9043 (2012).
[2] F. Höfling and T. Franosch, Rep. Prog. Phys. **76**, 046602 (2013).
[3] R. Metzler, J. H. Jeon, A. G. Cherstvy, and E. Barkai, Phys. Chem. Chem. Phys. **16**, 24128 (2014).
[4] Y. Meroz, I. M. Sokolov, and J. Klafter, Phys. Rev. E **81**, 010101(R) (2010).
[5] A. Iomin, V. Méndez, and W. Horsthemke, *Fractional Dynamics in Comb-like Structures* (World Scientific, Singapore, 2018).
[6] M. J. Saxton, Biophys. J. **81**, 2226 (2001).
[7] T. Kühn, T. O. Ihalainen, J. Hyväluoma, N. Dross, S. F. Willman, J. Langowski, M. Vihinen-Ranta, and J. Timonen, PLoS ONE **6**, e22962 (2011).
[8] K. Lindenberg, J. M. Sancho, M. Khoury, and A. M. Lacasta, Fluct. Noise Lett. **11**, 1240004 (2012).
[9] M. S. Simon, J. M. Sancho, and K. Lindenberg, Phys. Rev. E **88**, 062105 (2013).
[10] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[11] E. Barkai, R. Metzler, and J. Klafter, Phys. Rev. E **61**, 132 (2000).
[12] R. Metzler, E. Barkai, and J. Klafter, Phys. Rev. Lett. **82**, 3563 (1999).
[13] E. Barkai, Phys. Rev. E **63**, 046118 (2001).
[14] E. Lutz, Phys. Rev. E **64**, 051106 (2001).
[15] S. Burov and E. Barkai, Phys. Rev. E **78**, 031112 (2008).
[16] A. G. Cherstvy, A. V. Chechkin, and R. Metzler, New J. Phys. **15**, 083039 (2013).
[17] N. Leibovich and E. Barkai, Phys. Rev. E **99**, 042138 (2019).
[18] M. K. Lenzi, E. K. Lenzi, L. M. S. Guilherme, L. R. Evangelista, and H. V. Ribeiro, Physica A **588**, 126560 (2022).
[19] T. Sandev, V. Domazetoski, L. Kocarev, R. Metzler, and A. Chechkin, J. Phys. A: Math. Theor. **55**, 074003 (2022).
[20] S. C. Lim and S. V. Muniandy, Phys. Rev. E **66**, 021114 (2002).
[21] I. Eliazar, Physica A **658**, 130270 (2025).
[22] A. D. Viñales and M. A. Despósito, Phys. Rev. E **73**, 016111 (2006).
[23] G. H. Weiss, Physica A **311**, 381 (2002).
[24] J. Masoliver and G. H. Weiss, Eur. J. Phys. **17**, 190 (1996).
[25] O. Heaviside, *Electrical papers of Oliver Heaviside* (Chelsea New York, 1970), Vol. 1.
[26] M. Kac, Rocky Mountain J. Math. **4**, 497 (1974); V. Balakrishnan, Pramana J. Phys. **40**, 259 (1993).
[27] N. E. Ratanov, Markov Process. Relat. Fields **5**, 53 (1999).
[28] L. Angelani and R. Garra, Phys. Rev. E **100**, 052147 (2019).
[29] T. Sandev, L. Kocarev, R. Metzler, and A. Chechkin, Chaos Solitons Fractals **165**, 112878 (2022).
[30] T. Sandev and A. Iomin, Phys. Rev. E **110**, 024101 (2024).
[31] V. M. Kenkre, E. W. Montroll, and M. F. Shlesinger, J. Stat. Phys. **9**, 45 (1973).
[32] W. H. Haut Jr. and J. A. Buck, *Engineering Electromagnetics* 8th ed. (McGraw-Hill, New York, 2012).

[33] M. A. F. dos Santos, L. Menon Jr., and C. Anteneodo, Phys. Rev. E **106**, 044113 (2022); L. Menon Jr, M. A. F. dos Santos, and C. Anteneodo, J. Stat. Mech. (2023) 123203.

[34] T. Sandev and A. Iomin, *Special Functions of Fractional Calculus: Applications to Diffusion and Random Search Processes* (World Scientific, Singapore, 2022).

[35] A. Pal, V. Stojkoski, and T. Sandev, Random resetting in search problems, in "*The Target Problem*", edited by D. Grebenkov, R. Metzler, and G. Oshanin (Springer Nature, Cham, 2024).

[36] J. Masoliver and G. H. Weiss, Physica A **183**, 537 (1992); Phys. Rev. E **49**, 3852 (1994).

[37] J. Masoliver and K. Lindenberg, Eur. Phys. J. B **90**, 107 (2017).

[38] K. Górska, A. Horzela, E. K. Lenzi, G. Pagnini, and T. Sandev, Phys. Rev. E **102**, 022128 (2020).

[39] A. Compte and R. Metzler, J. Phys. A: Math. Gen. **30**, 7277 (1997).

[40] T. Kosztołowicz, Phys. Rev. E **90**, 042151 (2014).

[41] M. D'Ovidio and F. Polito, Theory Probab. Appl. **62**, 552 (2018).

[42] E. Awad and R. Metzler, Fract. Calc. Appl. Anal. **23**, 55 (2020).

[43] K. Górska, Phys. Rev. E **104**, 024113 (2021); K. Górska and A. Horzela, Fract. Calc. Appl. Anal. **26**, 480 (2023).

[44] K. Górska, F. J. Sevilla, G. Chacón-Acosta, and T. Sandev, Entropy **26**, 665 (2024).

[45] T. M. Michelitsch, F. Polito, and A. P. Riascos, Commun. Nonlinear Sci. Numer. Simul. **118**, 107031 (2023).

[46] T. M. Michelitsch, F. Polito, and A. P. Riascos, Mathematics **11**, 471 (2023).

[47] M. O. Cáceres, Phys. Rev. E **105**, 014110 (2022).

[48] M. O. Cáceres, M. Nizama, Phys. Rev. E **105**, 044131 (2022).

[49] J. Tailleur and M. E. Cates, Phys. Rev. Lett. **100**, 218103 (2008).

[50] F. Thiel, L. Schimansky-Geier, and I. M. Sokolov, Phys. Rev. E **86**, 021117 (2012).

[51] A. Villa-Torrealba, C. Chávez-Raby, P. de Castro, and R. Soto, Phys. Rev. E **101**, 062607 (2020).

[52] J. Saragosti, V. Calvez, N. Bournaveas, B. Perthame, A. Buguin, and P. Silberzan, Proc. Natl. Acad. Sci. USA **108**, 16235 (2011).

[53] S. Kamdar, S. Shin, P. Leishangthem, L. F. Francis, X. Xu, and X. Cheng, Nature (London) **603**, 819 (2022).

[54] R. N. Bearon and T. J. Pedley, Bull. Math. Biol. **62**, 775 (2000).

[55] B. Partridge, S. Gonzalez Anton, R. Khorshed, G. Adams, C. Pospori, C. Lo Celso, and C. F. Lee, PLoS ONE **17**, e0272587 (2022).

[56] D. S. Dean, S. N. Majumdar, and H. Schawe, Phys. Rev. E **103**, 012130 (2021).

[57] F. Mori, P. Le Doussal, S. N. Majumdar, and G. Schehr, Phys. Rev. Lett. **124**, 090603 (2020).

[58] P. Singh and A. Kundu, Phys. Rev. E **103**, 042119 (2021); J. Stat. Mech. **2019**, 083205 (2019).

[59] T. Bertrand, Y. Zhao, O. Bénichou, J. Tailleur, and R. Voituriez, Phys. Rev. Lett. **120**, 198103 (2018).

[60] C. Kurzthaler, Y. Zhao, N. Zhou, J. Schwarz-Linek, C. Devailly, J. Arlt, J. D. Huang, W. C. Poon, T. Franosch, J. Tailleur, and V. A. Martinez, Phys. Rev. Lett. **132**, 038302 (2024); Y. Zhao, C. Kurzthaler, N. Zhou, J. Schwarz-Linek, C. Devailly, J. Arlt, J.-D. Huang, W. C. K. Poon, T. Franosch, V. A. Martinez, and J. Tailleur, Phys. Rev. E **109**, 014612 (2024).

[61] L. Angelani, J. Phys. A: Math. Theor. **56**, 455003 (2023); Phys. Scr. **98**, 125013 (2023); L. Angelani and R. Garra, Modern Stochastics: Theory and Applications **6**, 3 (2018).

[62] H. C. Fogedby, Phys. Rev. E **50**, 1657 (1994).

[63] A. V. Chechkin, M. Hofmann, and I. M. Sokolov, Phys. Rev. E **80**, 031112 (2009).

[64] I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego 1999).

[65] L. Angelani, A. De Gregorio, R. Garra, and F. Iafrate, J. Stat. Phys. **191**, 129 (2024).

[66] K. Malakar, V. Jemseena, A. Kundu, K. V. Kumar, S. Sabhapandit, S. N. Majumdar, S. Redner, and A. Dhar, J. Stat. Mech. **2018**, 043215 (2018).

[67] M. R. Evans and S. N. Majumdar, J. Phys. A: Math. Theor. **51**, 475003 (2018).

[68] J. Masoliver, Phys. Rev. E **99**, 012121 (2019).

[69] P. Jolakoski, P. Trajanovski, A. Iomin, L. Kocarev, and T. Sandev, Europhys. Lett. **149**, 41004 (2025).

[70] V. V. Uchaikin and R. Sibatov, *Fractional Kinetics in Solids: Anomalous Charge Transport in Semiconductors, Dielectrics, and Nanosystems* (World Scientific, Singapore, 2013).

[71] M. R. Evans and S. N. Majumdar, J. Phys. A: Math. Theor. **47**, 285001 (2014).

[72] A. Masó-Puigdellosas, D. Campos, and V. Méndez, Phys. Rev. E **99**, 012141 (2019).

[73] A. S. Bodrova, A. V. Chechkin, and I. M. Sokolov, Phys. Rev. E **100**, 012120 (2019); **100**, 012119 (2019).

[74] M. R. Evans, S. N. Majumdar, and G. Schehr, J. Phys. A: Math. Theor. **53**, 193001 (2020).

[75] M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. **106**, 160601 (2011).

[76] A. Pal, A. Kundu, and M. R. Evans, J. Phys. A: Math. Theor. **49**, 225001 (2016).

[77] A. S. Bodrova and I. M. Sokolov, Phys. Rev. E **101**, 052130 (2020); **101**, 062117 (2020).

[78] M. Radice, Phys. Rev. E **104**, 044126 (2021); J. Phys. A: Math. Theor. **55**, 224002 (2022).

[79] O. Tal-Friedman, Y. Roichman, and S. Reuveni, Phys. Rev. E **106**, 054116 (2022).

[80] C. Di Bello, A. V. Chechkin, A. K. Hartmann, Z. Palmowski, and R. Metzler, New J. Phys. **25**, 082002 (2023).

[81] A. Pal and V. V. Prasad, Phys. Rev. E **99**, 032123 (2019); C. Christou and A. Schadschneider, J. Phys. A: Math. Theor. **48**, 285003 (2015).

[82] G. Tucci, A. Gambassi, S. Gupta, and É. Roldán, Phys. Rev. Res. **2**, 043138 (2020).

[83] D. Das and L. Giuggioli, J. Phys. A: Math. Theor. **55**, 424004 (2022).

[84] P. Julián-Salgado, L. Dagdug, and D. Boyer, Phys. Rev. E **109**, 024134 (2024).

[85] V. Méndez, R. Flaquer-Galmés, and D. Campos, Phys. Rev. E **109**, 044134 (2024).

[86] C. Fox, Trans. Amer. Math. Soc. **98**, 395 (1961).

[87] B. L. J. Braaksma, Compositio Mathematica **15**, 239 (1962–1964).

[88] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-function: Theory and Applications* (Springer, Berlin, 2009).

[89] T. R. Prabhakar, Yokohama Math. J. **19**, 7 (1971).

[90] R. Garra and R. Garrappa, Commun. Nonlin. Sci. Num. Simul. **56**, 314 (2018).

[91] W. Feller, *An Introduction to Probability Theory and its Applications* (John Wiley & Sons, New York, 2008), Vol 2.